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Mean-VaR Portfolio Optimization Based on the Improved Knapsack Problem: Parametric and Nonparametric Approaches

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Abstract

One of the most important problems in portfolio selection models is the ability to provide the optimal number of each share. Therefore, in some cases, it interferes with portfolio optimization in converting the desired weight per share to the desired number per share, unless the results are an integer. Moreover, by applying the appropriate strategy, it seems possible to discover the optimal stock allocation for significant cases with comparatively large stock value. In this regard, this study presents a multi- objective portfolio selection model considering cardinality, quantity and budget constraints based on a new improved knapsack problem. Value-at-Risk (VaR) is considered as the second objective function of risk assessment in the knapsack-based portfolio selection model. We consider parametric (variance- covariance matrix) and non-parametric (historical) approaches to measure VaR. The study also uses the best GARCH family models to estimate the conditional volatility of return in the variance-covariance matrix, which is based on measuring and comparing different criteria under various types of GARCH family models.
Finally, a Non-dominated Sorting Genetic Algorithm II (NSGA II) is planned to solve the problem. An actual portfolio of the Iran stock market is solved to demonstrate the application of the suggested model.

Highlights

- A multi-objective portfolio selection model based on the improved knapsack problem is proposed.
- The proposed model can properly allocate the number of shares to different assets.
- The proposed model is very suitable for the value of a particular share becomes relatively large.
- Results show the superiority of the proposed model and the importance of selecting the appropriate method for risk measurement in portfolio optimization problems.

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1. Introduction

Markowitz (1952, 1959) presented a classical mean-variance model and assumed that investors would like to maximize the expected return for an appointed to some extent of risk or minimize the risk of the portfolio for an appointed expected return. One of the most important and controversial points in portfolio optimization models is the type of decision variables used in these models. When the type of decision variables in the portfolio optimization models is continuous variables, portfolio optimization is involved with the optimal stock allocation for significant cases with comparatively large stock value. In order to get rid of this weakness, this research focuses on providing a portfolio optimization model based on the discrete decision variables. According to the dual Lagrangian relaxation method and transformation approaches, Li and Tsai (2008) studied a discrete portfolio optimization problem. Bonami and Lejeune (2009) presented a nonlinear branch and bound algorithm for the portfolio selection model with discrete variables. Anagnostopoulos and Mamanis (2010) investigated a portfolio selection model with three-objective, class and quantity limitations under nonlinear mixed-integer programming using various multi-objective evolutionary algorithms. Several test sets have been used to study discrete portfolio optimization by Castro et al. (2011).

In this study, when the cost of purchasing a particular asset becomes relatively large, we concentrate on the asset allocation problem, e.g. Berkshire Hathaway Inc. Therefore, the portfolio optimization problem is presented on the basis of the improved knapsack problem due to the cardinality, quantity and budget constraints. Also, Value-at-Risk (VaR) is considered as the second objective function for risk assessment in the knapsack-based portfolio selection model. (variance- covariance matrix) and non-parametric (historical) approaches are used to measure VaR and the results of these approaches are compared with each other. The best model of GARCH family models is used to estimate the conditional volatility of return in the variance- covariance matrix, which is based on measuring and comparing different criteria in different types of GARCH family models. Therefore, a Multi-Objective Evolutionary Algorithm (MOEA) is planned to solve the problem. Real portfolio of the Iran stock market has been solved to demonstrate the application of the suggested model.

Therefore, we have introduced a multi-objective portfolio selection model on the basis of the improved knapsack problem taking into account the cardinality, quantity and budget constraints. The proposed model is based on discrete variables which permit the manner of some portfolio optimization problems in a more down-to-earth approach and present the probability of adding some pure specifications to the model. Moreover, using aknapsack type optimization strategy, it seems possible to discover the optimal stock allocation for significant cases with comparatively large stock value.

The first objective function maximized the expected return of the portfolio, and the second objective function minimized VaR of the portfolio for risk assessment in the knapsack-based portfolio selection model. By comparing the results of two sets of objective functions based on different approaches in risk measurement, a superior approach in risk assessment is revealed for the knapsack-based portfolio selection model.

Consequently, Pareto optimal frontiers of the mean- parametric VaR portfolio selection model were superior to the Pareto optimal frontiers of the mean- nonparametric VaR portfolio selection model. These results show the importance of selecting the appropriate method for risk measurement in portfolio optimization problems.

The structure of the paper is as follows. In section 2, an overview of the VaR, different types of GARCH family models and NSGA II are presented. According to the knapsack problem, the multi-objective portfolio optimization problem considering the cardinality, quantity and budget constraints is presented in section 3. Section 4 reports the computational results of the numerical example. Eventually, the main results are presented in section 5.

2. Literature Review

In the Markowitz's model, the investment risk was measured by the variance of the return. Variance is recognized as a symmetric risk evaluation case and has been pilloried by numerous scholars. In addition, estimating the variance error reduces the quality of the capital distribution. Among the various methods of risk measurement, VaR is one of the most popular alternative methods (Jorion, 1997).

Some studies focus on combined VaR-based approaches. Combination of copula functions and Generalized Auto-Regressive Conditional Heteroscedasticity (GARCH) model has been used to estimate a mean-VaR portfolio optimization problem (Huang et al., 2009) and the Markov switching approach has been used to examine a portfolio optimization problem with VaR constraint (Yiu et al., 2010; Zhu et al., 2016).

Other studies have compared different methods of measuring VaR. For example, Ranković et al. (2016) presented a mean- univariate GARCH VaR based on the Markowitz theorem and compared its results with the mean- historical VaR model. Banihashemi and Navidi (2017) introduced a mean- conditional Value-at-Risk (CVaR) model and compared its results with the mean- VaR portfolio optimization using Data Envelopment Analysis. Meghwani and Thakur (2018) and Guo et al. (2019) presented cardinality constrained portfolio optimization problem under several risk measures such as variance, VaR and CVaR and compared them. Besides, Lwin et al. (2017) compared five meta-heuristic algorithms for mean-VaR Markowitz model.

2.1 Review of Literature on VaR

Among the various types of risk assessments, VaR (Morgan, 1996) is a very common method for risk measurement in financial markets. VaR measures the worst expected loss of a portfolio over a scheduled period at a given confidence level. VaR is defined in the Eq. (1):

$$Pr\left[R_t < -VaR_t(\alpha)\right] = \alpha, \tag{1}$$

where R_t is the portfolio return at time t and α is the confidence level. $VaR_t(\alpha) = -F_{R_t}^{-1}(\alpha)$, $F_{R_t}^{-1}(\alpha)$ is the cumulative distribution function of returns.

Different methods are used to compute VaR such as the parametric approach (variance–covariance matrix), the nonparametric approach (historical simulation) (Pritsker, 2006), and Monte Carlo simulation methods (Linsmeier & Pearson, 2000). In parametric estimation, it is assumed that the financial returns follows a probability distribution function such as a normal distribution function with the parameters μ, σ^2 . In the nonparametric approach, there is no assumption about the distribution function. However, it is assumed that the financial returns behavior is the same as its past behavior. Monte Carlo simulation using statistical sampling and random scenarios provides approximate answers to quantitative problems. Pagan and Schwert (1990) studied the performance of parametric and nonparametric methods. The application of these methods is greatly influenced by the needs of analysts, decision-making authorities of the organization, the type of assets under study, the accuracy and speed of the calculations and other considerations.

The historical approach is a non-parametric method based on previous information. This approach assumes that the near future is closely related to the past so that past information can be used to predict future risks (Alexander, 2009). To estimate VaR, it is enough to extract the α percentile of the return distribution. To better define this approach, $\Pi_t(X)$ is defined in the Eq. (2):

$$\Pi_t(X) = \sum_{i=1}^{n} x_i r_{it}, \quad t = 1,2,\ldots,T \tag{2}$$

where $\Pi_t(X)$ is the portfolio return under scenario, t, r_{it} is the observed return of asset i at time t, x_t denotes the number of asset i $(i \in \{1,2,\ldots,n\})$, and T represents the time series length.

Therefore, the total return of the portfolio is in the Eq. (3):

$$E(R_P) = \sum_{t=1}^{T} \Pi_t(X)\rho_t, \tag{3}$$

where ρ_t is the probability of scenario t. We assume that the probability of all scenario are equal to $\rho_t = 1/T$. Therefore, the VaR based on the α percentile of the return distribution using the historical data can be formulated in the Eq. (4):

$$VaR_{t+1}^{\alpha}(R_P) = inf\left\{\Pi_t(X) \mid \sum_{t=1}^{T}\rho_t \geq \alpha\right\}. \tag{4}$$

It should be noted that returns should be sorted ascending $\left(\Pi_1(X) \le \Pi_2(X) \le \ldots \le \Pi_T(X)\right)$.

On the other hand, the analytical VaR is a parametric method that its return is defined in the Eq. (5):

$$R_t = \mu_t + \varepsilon_t \,, \tag{5}$$

where R_t is the return at time t, μ_t is the conditional mean of the return $\left(\mu_t = E\left(R_t \mid \Omega_{t-1}\right)\right)$, and Ω_{t-1} is the information of the previous period). ε_t is the return shock ($E\left(\varepsilon_t \mid \Omega_{t-1}\right) = 0,$ ($E\left(\varepsilon_t^2 \mid \Omega_{t-1}\right) = \sigma_t^2$, $\varepsilon_t \approx D\left(0, \sigma_t^2\right)$) and the more detailed definition of ε_t is written in the Eq. (6):

$$\varepsilon_t = \sigma_t v_t \,, \tag{6}$$

where in σ_t is the conditional volatility of the return and v_t is the innovation sequence $\left(E\left(v_t\right) = 0, E\left(v_t^2\right) = 1, v_t \approx iid\left(0{,}1\right)\right)$. All models can be estimated with the assuming a normal distribution or a Student t distribution. The maximum likelihood estimation of the normal distribution can be formulated in the Eq.(7):

$$L = \sum_{t=\varsigma}^{T}\left(-\frac{1}{2}ln\left(2\Pi\right) - \frac{1}{2}ln\left(\sigma_t^2\right) - \frac{1}{2}\frac{\varepsilon_t^2}{\sigma_t^2}\right), \tag{7}$$

where in ς is the number of observations that are lost in the estimation process. The maximum likelihood estimation of the Student t distribution can be formulated in the Eq. (8):

$$L = -\sum_{t=\varsigma}^{T}\left[\frac{\tau+1}{2}ln\left(1 + \frac{\varepsilon_t^2}{\left(\tau-2\right)\sigma_t^2}\right) + \frac{1}{2}ln\left(\sigma_t^2\right)\right], \tag{8}$$

where in τ is the degrees of freedom. Since the financial returns typically maintain wider sequences, the standardized Student t distribution better describes their characteristics. Therefore, VaR is defined in the Eq. (9):

$$VaR_{t+1}^{\alpha} = \mu_t h + \sigma_t\, t_{\alpha}^{-1}\left(v\right)\sqrt{h}, \tag{9}$$

where μ_t is the conditional mean of the return, h is the time horizon, $t_{\alpha}^{-1}\left(v\right)$ is $\alpha - quantile$ of the standardized Student t distribution $\left(t\left(v\right) \approx iid\left(0{,}1\right)\right)$ with v degrees of freedom and σ_t is the conditional volatility of the return. In this study, σ_t is estimated using GARCH family models. In estimating GARCH parameters, the conditional mean is dominated by the standard deviation of returns (see Ranković et al., 2016; Pritsker, 2006; Alexander, 2009; Christoffersen, 2011). This implies:

$$VaR_{t+1}^{\alpha} = \sigma_t \, t_{\alpha}^{-1}(\nu)\sqrt{h}. \tag{10}$$

In order to compute the VaR for a multiple asset's portfolio $\left(X' = (x_1, x_2, \ldots, x_n)\right)$, H is defined as the conditional variance– covariance matrix of the returns. Thus, the variance can be written more compactly in the Eq. (11):

$$\sigma_P^2 = X'HX \tag{11}$$

Therefore, the VaR of a multiple asset's portfolio can be obtained in the Eq. (12):

$$VaR_{t+1}^{\alpha}(X) = \left(X'H_{t+1}X\right)^{1/2} t_{\alpha}^{-1}(\nu)\sqrt{h}. \tag{12}$$

In this study, the conditional variance–covariance matrix of the returns (H) is calculated based on the estimation of GARCH family models.

2.2 Review of Literature on GARCH Family Models

Among the various types of time series models, GARCH family models have had successful performance in modeling and forecasting conditional volatility of return. Engle (1982) presented the Autoregressive Conditional Heteroskedastic (ARCH) model. The main idea of this model is that the return shocks do not have a serial correlation, but are nonlinearly related to each other, which can be explained by a quadratic function. The ARCH (q) model can be formalized in the Eq. (13):

$$\sigma_t^2 = \varphi_0 + \sum_{i=1}^{q} \varphi_i \varepsilon_{t-i}^2 \ , \tag{13}$$

where $\varphi_0 > 0, \varphi_i \geq 0$ and σ_t^2 is the conditional variance of ε_t.

Note that the ARCH model typically requires estimating a large number of parameters. Moreover, a specific and predetermined structure must be applied to avoid the negative values of the estimated variance. Therefore, Bollerslev (1986) extended the Engle model and presented a group of models known as Generalized ARCH (GARCH) model. The GARCH (p, q) model can be formulated in the Eq. (14):

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \lambda_j \sigma_{t-j}^2 \ , \tag{14}$$

where $\omega > 0, \beta_i \geq 0, \lambda_j \geq 0$, and $\sum_{i=1}^{q} \beta_i + \sum_{j=1}^{p} \lambda_j < 1$.

For example, GARCH (1, 1) is a special branch of the GARCH model, which has an acceptable ability to examine financial time series (see So & Philip, 2006). GARCH (1, 1) can be formulated in the Eq. (15):

$$\sigma_t^2 = \omega + \beta\varepsilon_{t-1}^2 + \lambda\sigma_{t-1}^2. \quad (15)$$

Based on the general GARCH model, several models have been extended that emphasize on specific features of financial data. For instance, in the Integrated GARCH (IGARCH) model, the Eq. (16) is imposed on coefficients (see Engle & Bollerslev, 1986).

$$\sum_{i=1}^{q}\beta_i + \sum_{j=1}^{p}\lambda_j = 1. \quad (16)$$

One of the limitations in the GARCH model is that the effects of positive and negative shocks are considered, symmetrically. Therefore, Nelson (1991) presented the Exponential GARCH (EGARCH) model. The EGARCH (p, q) model can be formalized in the Eq. (17):

$$ln\ \sigma_t^2 = \omega + \sum_{j=1}^{p}\lambda_j\ ln\ \sigma_{t-j}^2 + \sum_{i=1}^{q}\xi_i\left(\left|\frac{\varepsilon_{t-i}}{\sigma_{t-i}} - \frac{\sqrt{2}}{\Pi}\right|\right) + \sum_{i=1}^{q}\beta_i\ \frac{\varepsilon_{t-i}}{\sigma_{t-i}}. \quad (17)$$

Glosten, Jagannathan, and Runkle (GJR) -GARCH is another asymmetric model presented by Glosten et al. (1993). The GJR-GARCH (p, q) model can be formulated in the Eq. (18):

$$\sigma_t^2 = \omega + \sum_{i=1}^{q}\beta_i\varepsilon_{t-i}^2 + \sum_{j=1}^{p}\lambda_j\sigma_{t-j}^2 + \sum_{i=1}^{q}\xi_i I_{t-i}\varepsilon_{t-i}^2\ , \quad (18)$$

where I_{t-i} is the difference between positive and negative shocks, If $\varepsilon_{t-i} \geq 0$ then $I_{t-i} = 0$ else $I_{t-i} = 1$. Also, $\omega > 0, \beta_i \geq 0, \lambda_j \geq 0, \xi_i \geq 0$, and $\sum_{i=1}^{q}\beta_i + \sum_{j=1}^{p}\lambda_j + \frac{1}{2}\sum_{i=1}^{q}\xi_i < 1$.

Finally, more information on GARCH family models can be found in Guidolin and Pedio (2018). To use GARCH family models, the time series should pass the Augmented Dickey-Fuller (ADF) test (Dickey & Fuller, 1979) , Ljung–Box Q test and the ARCH-LM test (Engle, 1982). Accordingly, the optimal model is selected based on the minimum Akaike Information Criteria (AIC) (Akaike, 1974), the minimum Schwarz Information Criteria (SIC) (Schwarz, 1978) and the minimum Hannan–Quinn Information Criteria (HQIC) (Hannan & Quinn, 1979).

The Akaike Information Criterion (AIC):

$$AIC = 2k - 2\log\left(L_{max}\right). \quad (19)$$

The Schwarz Information Criterion (SIC):

$$SIC = \ln\left(n\right)k - 2\ln\left(L_{\max}\right). \quad (20)$$

The Hannan-Quinn Information Criterion (HQIC):

$$HQIC = 2\ln\left(\ln\left(n\right)\right)k - 2\ln\left(L_{\max}\right), \tag{21}$$

Where in k is the number of parameters, n is the number of observations and L_{max} is the maximum likelihood estimation.

Figure 1 describes the steps and how to choose the best model among the GARCH family models:

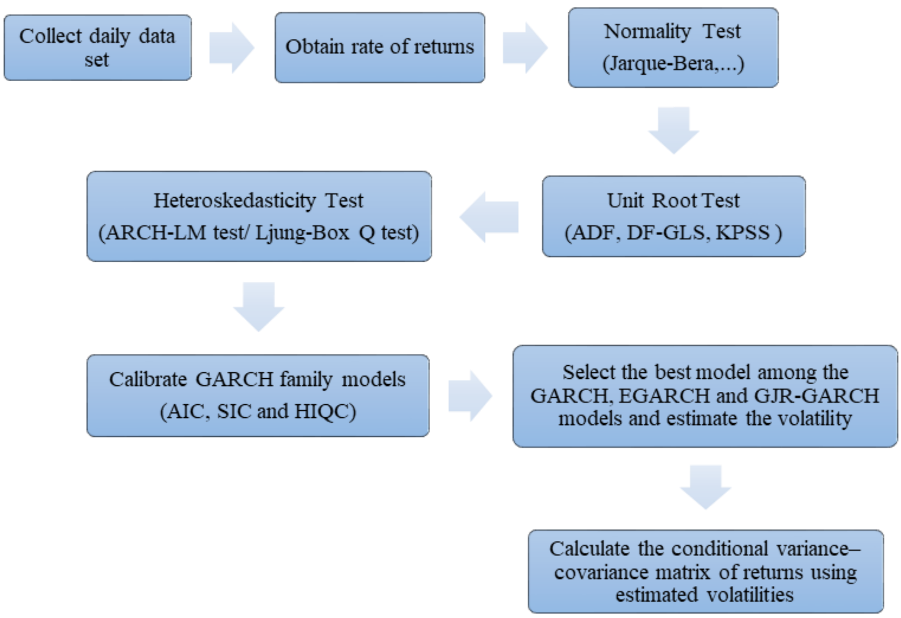


Figure 1. Framework of the approach.
***Source**: Engle (1982) and Bollerslev (1986)*

2.3 Review of Literature on NSGA-II

Portfolio optimization on the basis of the knapsack problem is NP-complete (Böckenhauer et al., 2012; Kellerer et al., 2004). Thus, a meta-heuristic method is applied. Multi-objective genetic algorithm is a population-based algorithm and is very proper for solving multi-objective optimization problems (Fonseca & Fleming, 1993) . One of the multi-objective optimization methods on the basis of the genetic algorithm is NSGA-II, which is presented by Deb et al. (2002). In fact, this algorithm is an improved NSGA introduced by Srinivas and Deb (1994). In this approach, there are three main concepts including: (I) dominance; (II) non-dominated sorting; and (III) a diversity of solutions that form the basis of multi-objective optimization.

(I) The concept of domin ance can be expressed below. In a minimization problem with more than one objective function, the point X dominates point Y if and only if Y has no better aspect than X and at least one aspect of X is much better than Y. Points that meet these conditions are considered the first front (Fonseca & Fleming, 1995) . Assume that the objective functions are in Eq. (22):

$$\begin{cases} Minimize\left\{ f_1\left(X\right), f_2\left(X\right), ..., f_n\left(X\right)\right\} \\ s.t: g\left(X\right) \leq 0, \end{cases} \tag{22}$$

where X is the vector of the decision variable ($\{x_1, x_2, ..., x_n\}$) and $f_i\left(X\right)$ is the objective function i. Therefore, the concept of dominance can be formulated in Eq. (23) and Eq. (24):

$$f_i\left(X\right) \leq f_i\left(Y\right), \quad for\ all\ i \in \{1, 2, ..., n\} \tag{23}$$

$$f_i\left(X\right) < f_i\left(Y\right), \quad for\ at\ least\ one\ i \in \{1, 2, ..., n\}. \tag{24}$$

(II) In a multi-objective optimization problem with at least two objective functions, the concept of non-dominated sorting of solutions can be expressed in the following. In many cases, we cannot compare the solutions with the concept of dominance. In fact, some solutions may not be dominated other and it may not be easy to make a definitive decision about some solutions. Therefore, to get the best solutions, they should be sorted according to specific criteria. In this algorithm, each solution is assigned a rank based on the number of times the solution dominated other solutions. At the end of the algorithm, that have the best-ranked solutions (front 1) are selected as the answer set or the Pareto front (see Rey Horn et al., (1993)).

(III) Sometimes the members of an answer set have the same rank. In these cases, the solutions are compared and some are removed. The concept of diversity is used to remove some members of the answer set. In this approach, solutions that are regular in each interval of the answer set are selected. This process is based on the crowding distance operator which can be formulated in Eq. (25) and Eq. (26):

$$Cd_i^j = \frac{\left| f_{i+1}^j - f_{i-1}^j \right|}{f_{\max}^j - f_{\min}^j}, \tag{25}$$

$$Cd_i = \sum_{j=1}^{n} Cd_i^j, \tag{26}$$

where Cd_i^j is crowding distance i of objective function j . Cd_i is the total crowding distance of all objective functions. f_{i+1}^j and f_{i-1}^j are the values of objective functions j in solution $i+1$ and $i-1$, respectively. In addition, $f_{\max}^j$ is the maximum value of the objective function j and $f_{\min}^j$ is the minimum value of the objective function j.

Any point that has a greater crowding distance means that it covers more range of the solution space, and its removal leads to the loss of solution diversity in a wide range of solution space. Therefore, the points of the solution set in a frontier with less crowding distance should be eliminated until the initial population remains constant. In addition, the starting and ending points of this set are important points that must necessarily be among the solutions (Deb et al., 2002, 2000).

In addition, there are other general sections such as, (IV) calculating the objective functions and (V) creating an initial population. (VI) Another part of this algorithm is associated with two types of operators for generating parent and offspring. In other words, to create diversity in the population, recombine the pairs of parent and mutate the offspring. Usually, the probability of recombining the parent pairs or the crossover rate $\left(p_c\right)$ is chosen between 0.6 and 0.9 and the probability of mutating the offspring or the mutation rate $\left(p_m\right)$ is chosen between 0.05 and 0.2.

Now, according to the mentioned definitions, the NSGA-II steps can be described in Algorithm 1:

Algorithm 1. Pseudocode of NSGA-II

Assume that $f_1(X), f_2(X)$ are the objective functions of

$X = \left(x_1, x_2, ..., x_n\right)^T$;

Give value for p_c, p_m and $Max\,Iteration$;

Create initial population randomly (N_0);

Evaluate objective functions;
Assign rank using fast non-dominated sorting;
Recombine pairs of parents;
Mutate the offspring;
while $(t \leq Max\,Iteration)$ **do**
  for $(i = 1 : Number\,Generation)$ **do**
    Assign rank using fast non-dominated sorting;
    Build set of non-dominated solution;
    Determine crowding distance;
  End
  For $(Length\,of\;population \leq N)$ **do**
    Build the current Pareto front based on the lower front and high crowding distance;
  End
  $t = t + 1$;
  Recombine and Mutate;
End
Show result and visualization.

Source: *Deb et al. (2002).*

3. Proposed Model

In order to easily explain the mean-VaR portfolio optimization problem on the basis of the improved knapsack problem, the following symbols are applied:

n,	**the whole number of available assets;**
k,	the desired number of assets that can be selected in the portfolio;
B,	the whole budget;
$\overline{p}_i$,	the average price of asset i during the period;
u_i,	the upper bound of asset i;
l_i,	the lower bound of asset i;
α,	the confidence level;
$VaR_{t+1}^{\alpha}\left(R_P\right)$,	the estimated VaR of a portfolio at confidence level α;
$E\left(R_P\right)$,	the expected return of a portfolio;
x_i,	the integer variable that illustrate the number of asset i;
y_i,	the binary variable specifying whether asset i is contained in a portfolio or not. $y_i = 1$, if asset i is contained in a portfolio, and $y_i = 0$ differently.

The multi-objective portfolio selection model based on the improved knapsack problem can be formulated in Eq. (27) to Eq. (33):

$$
\left(M_1\right)\begin{cases}
\underset{X}{\max imize}\ E\left(R_P\right) & (27)\\
\underset{X}{\min imize}\ VaR_{t+1}^{\alpha}\left(R_P\right) & (28)\\
s.t: & \\
\sum_{i=1}^{n}\overline{p}_i x_i \le B, & (29)\\
\sum_{i=1}^{n} y_i = k, & (30)\\
l_i y_i \le x_i \le u_i y_i\ , \forall i \in \{1,2,...,n\}, & (31)\\
x_i \in int, i \in \{1,2,...,n\}. & (32)\\
y_i \in \{0,1\}, i \in \{1,2,...,n\}. & (33)
\end{cases}
$$

The target function given in Eq. (27) tries to choose a portfolio that has the maximum expected returns. Also, the objective function given in Eq. (28) attempts to measure the worst expected loss of a portfolio. VaR is measured according to the following approaches: Parametric approach

In the parametric approach, the variance-covariance matrix is used to estimate the VaR. Therefore, Eq. (28) is defined according to Eq. (12) $VaR_{t+1}^{\alpha}\left(X\right)=\left(X'H_{t+1}X\right)^{1/2}t_{\alpha}^{-1}\left(\nu\right)\sqrt{h}$. for more accurate and better estimates, GARCH family models are used to estimate GARCH family models that have performed successfully in modeling and forecasting the conditional volatility of return. Thus, this model is called the mean-GARCH (the best model of GARCH family models) VaR portfolio optimization problem on the basis of the knapsack problem.

1) nonparametric approach

In the non-parametric approach, historical simulation is used to estimate the VaR. Therefore, Eq. (28) is defined according to Eq. (4) $(VaR_{t+1}^{\alpha}\left(R_P\right)=inf\left\{\Pi_t\left(X\right)|\sum_{t=1}^{T}\rho_t\geq\alpha\right\})$. Thus, this model is called the mean-historical VaR portfolio optimization problem on the basis of the knapsack problem.

Finally, the results of solving the proposed selection model are compared with each other and the best risk measurement approach of the proposed model is introduced.

Eq. (29) defines the budget constraints that states that the sum of the weights of the total assets is less than or equal to the financial plan. Eq. (30) defines the cardinality constraint which guarantees that a portfolio contains exactly k assets. $y_i=1$, if asset i is involved in a portfolio, and $y_i=0$ otherwise. Eq. (31) defines the floor and ceiling constraints. If asset i is selected ($y_i=1$), the proportion of capital x_i lies in $\left[l_i,u_i\right]$.

Eq. (32) and Eq. (33) define the decision variables. $x_i(\forall i\in\{1,2,\ldots n\})$ is the integer variable that represents the number of asset i. $y_i(\forall i\in\{1,2,\ldots n\})$ is the binary variable indicating whether asset i is involved in a portfolio or not.

The decision variables of the Markowitz model are continuous variables, while the decision variables of the knapsack-based portfolio optimization are binary (Sahni, 1975) and (or) integer (Kellerer et al., 2004) variables. Therefore, the knapsack problem is more suitable for asset allocation in some cases compared to Markowitz theorem. Besides, using the knapsack type optimization approach, we determine the optimal asset allocation in certain specific cases with comparatively immense stock value. For example, suppose a portfolio consists of three assets with three weights of 0.5, 0.3 and 0.2, and we intend to assign one million dollars for these three shares with market prices of 1200$, 1800$ and 340,000$, respectively. Thus, the number of shares is 416.666, 166.666 and 0.588, respectively. As we can observe, 416.666 and 166.666 are non-integer number and the third value does note yield a feasible allocation of asset (Vaezi et al. (2019, 2020). In the following, the different aspects and benefits of the suggested model are explained.

- The suggested model maximizes the expected returns on the portfolio and minimizes the worst expected loss to the portfolio, simultaneously.
- The mean-VaR portfolio optimization problem on the basis of the improved knapsack problem, considering the cardinality, quantity and budget constraints can decently assign the number of shares to various assets.
- In the suggested model, Parametric (variance-covariance matrix) and nonparametric (historical) approaches is considered to measure VaR.
- The best model of GARCH family models is used to estimate the conditional volatility of return in the variance-covariance matrix, which is based on measuring and comparing different criteria in different types of GARCH family models. The study also concentrates on the multi- objective portfolio optimization with discrete variables to remove the gap of share rounding.
- The proposed model is based on discrete variables that permit the manner of some portfolio optimization problems in a more down-to-earth approach and present the probability of adding some pure specifications to the model.
- Moreover, using a knapsack type optimization strategy, it seems feasible to discover the optimal stock allocation for significant cases with comparatively large stock value.

4. Empirical Analysis

In this section, an actual portfolio is provided to explain the performance of the proposed model. The numerical example involves the shares of the Iranian stock market shares and includes daily time series data $(h=1)$ from 2٦/٥/2016 to ٢٦/٥/2021; totaling 1000 trading days.

Let $R_t = (\ln(p_t) - \ln(p_{t-1})) \times 100$ denote the continuously compounded rate of asset returns (see Morelli, 2002), where p_{t-1} and p_t are the asset prices at time $t-1$ and t, respectively. In addition, the volatilities of the returns were estimated using GARCH family models. The following process should be used to select the appropriate model among the GARCH family models:

The first step in this process is to examine whether the distribution of the Iran stock market returns is normal or not and to have an overview of the basic statistical features for each of the Iranian stock market returns series is essential. Therefore, table 1 presents the average, median, maximum, minimum, standard deviation, skewness, kurtosis, and JB information for each of the series of the Iranian stock market returns.

Finally, the distance between the maximum and minimum value of the returns in each time series and their standard deviation causes relatively large fluctuations in the series of returns of the Iranian stock market in the sample period. Also, the amount of kurtosis in each return series is much higher than the standard value of the normal distribution (+3). This indicates that the distribution of the return series of Iranian stock market has the characteristics of ‘sharp peak’ and ‘fat tail’. In addition, its JB stats for each return series are much higher than the JB value of the standard normal distribution. Therefore, the null hypothesis that the return series is subject to a normal distribution is rejected. As a result, , the Iranian stock market returns series do not have a normal distribution and have Skewness (Panel A).

The second stage is the static study of the Iranian stock market return series with the unit root test statistics. The Augmented Dickey-Fuller (ADF) and KPSS are used for all of the time series. The results show that the ADF and KPSS values of the whole yield series are much smaller than 1% of the critical value and the P-values are 0 $\left(p \leq 0.05\right)$, also indicative of the significance at the 1% level. Therefore, all series are at a fixed level (Panel B).

Often, when residuals are correlated with each other over time, there are clustering fluctuations in financial time series. Hence, if conditional variance is assumed as a self-correlated function and is affected by previous residuals, modeling the clustering volatility is performed. In fact, this model allows the effect of a shock not to disappear quickly over time (see Dana, 2016). Therefore, the third step in this process is to examine the effects of ARCH in each return series of Iranian stock market. The ARCH-LM test is applied to examine the effects of ARCH. To estimate regression for the ARCH-LM test, the lag order is determined based on the smallest AIC and the largest F statistic (see Gökbulut & Pekkaya, 2014; Mamipour & Vaezi Jezeie, 2015). Therefore, the optimal lag order is 1 (ARCH (1)) for all of the return series. In addition, the optimal lag order of ARCH model is obtained using Correlogram of Residuals Squared (Ac and PAC). Therefore, the optimal lag order is 1 for all of the return series. Finally, the results show that the value of the F or R-square statistic is less than 5% of the critical value. Therefore, the null hypothesis of the ARCH-LM test must be rejected. This means that the F and R square statistics confirm the effects of ARCH in 20 series of returns out of 32 series of returns of the Iran stock market. Now, we can use the GARCH family models for estimating the conditional return fluctuations (Panel C).

Note that the estimates will not be accurate and efficient, if the above are not checked for using GARCH family models. Therefore, the results will not be valid. All results are presented in Table 1.

Table1. Descriptive statistics, unit roots tests and ARCH-LM tests

	R_{S1}	R_{S2}	R_{S3}	R_{S4}	R_{S5}	R_{S6}	R_{S7}
Panel A: Descriptive statistics							
Observations	1000	1000	1000	1000	1000	1000	1000
Mean	0.194586	0.249327	0.279560	0.344201	0.136960	0.253335	0.315650
Median	-0.065650	0.028700	0.005550	0.047850	-0.063050	0.000000	0.059650
Maximum	16.27470	17.86160	17.30090	11.19210	44.80910	17.26990	17.89760
Minimum	-10.05980	-10.17510	-9.940200	-7.203100	-8.861100	-5.129300	-5.129300
Std. Dev.	2.512264	2.737893	2.787408	2.428766	2.846338	2.543499	2.469925
Skewness	0.294392	0.329995	0.308139	0.222108	4.026733	0.380104	0.613941
Kurtosis	5.122603	5.050061	4.276992	3.483230	63.08498	4.763240	6.078457
JB	202.1713*	193.2641	83.77109	17.95165	153127.6	153.6221	457.6914
Panel B: Unit root test statistics							
ADF	-22.13651* (0.0000) **	-22.13651 (0.0000)	-21.84142 (0.0000)	-14.71584 (0.0000)	-23.05562 (0.0000)	-21.15619 (0.0000)	-14.74544 (0.0000)
KPSS***	0.187437	0.086574	0.081640	0.089483	0.207558	0.153307	0.081041
Panel C: ARCH-LM tests: ARCH (1)- Optimal lag:1							
F-statistic	22.95892* (0.0000) **	10.81843 (0.0010)	31.56339 (0.0000)	54.50891 (0.0000)	72.07124 (0.00001)	61.88803 (0.0000)	9.124355 (0.0026)
Obs*R-squared	22.48714* (0.0000) **	10.72377 (0.0011)	30.65618 (0.0000)	51.78692 (0.0000)	67.34740 (0.0001)	58.38780 (0.0000)	9.059745 (0.0026)

Table 1(continued). Descriptive statistics, unit roots tests and ARCH-LM test.

	R_{S8}	R_{S9}	R_{S10}	R_{S11}	R_{S12}	R_{S13}	R_{S14}
Panel A: Descriptive statistics							
Observations	1000	1000	1000	1000	1000	1000	1000
Mean	0.246704	0.265756	0.264079	0.235046	0.144038	0.164924	0.298804
Median	0.026700	-0.023150	0.000000	0.020700	-0.017950	-0.007900	0.005200
Maximum	15.48850	11.44370	15.10480	14.47630	9.716400	9.159600	20.99630
Minimum	-8.282800	-10.22770	-10.25140	-5.129000	-10.17340	-5.129300	-6.049300
Std. Dev.	2.407994	2.306047	2.355343	2.416607	2.206189	1.985585	2.378421
Skewness	0.242486	0.168991	0.312181	0.495790	0.142336	0.273030	0.861282
Kurtosis	4.968264	4.660646	5.177599	5.218276	4.354207	4.663851	9.771237
JB	171.2192	119.6657*	213.8235	245.9991	79.78808	127.7742	2034.036
Panel B: Unit root test statistics							
ADF	-21.61128 (0.0000)	-15.26661* (0.0000) **	-22.42845 (0.0000)	-22.42414 (0.0000)	-21.41658 (0.0000)	-20.77727 (0.0000)	-23.80032 (0.0000)
KPSS***	0.150785	0.189302	0.153309	0.135940	0.149653	0.163348	0.120051
Panel C: ARCH-LM tests: ARCH (1) -Optimal lag:1							
F-statistic	16.61804 (0.0000)	173.3152* (0.0000) **	57.49928 (0.0000)	24.95196 (0.0000)	231.4656 (0.0000)	243.6131 (0.0000)	5.573194 (0.0184)
Obs*R-squared	16.37838 (0.0001)	147.9447* (0.0000) **	54.47304 (0.0000)	24.39157 (0.0000)	188.2300 (0.0000)	196.1687 (0.0000)	5.553331 (0.0184)

Table 1(continued). Descriptive statistics, unit roots tests and ARCH-LM test.

	R_{S15}	R_{S16}	R_{S17}	R_{S18}	R_{S19}	R_{S20}
Panel A: Descriptive statistics						
Observations	1000	1000	1000	1000	1000	1000
Mean	0.155193	0.284443	0.240835	0.324616	0.361129	0.241878
Median	0.000000	0.000000	0.009750	-0.013500	0.000000	-0.028950
Maximum	13.36060	23.87580	9.145400	19.64680	22.74480	10.31080
Minimum	-37.48830	-12.27560	-5.821000	-7.956900	-13.87390	-16.60240
Std. Dev.	2.457393	2.947481	2.029996	2.735206	2.299747	2.938791
Skewness	-3.238543	0.687062	0.287326	0.502565	1.113789	-0.270471
Kurtosis	58.05501	7.606584	4.366210	5.688044	14.19757	4.412281
JB	128041.9	962.8681	91.53136*	343.1611	5431.157	95.29810
Panel B: Unit root test statistics						
ADF	-13.52366 (0.0000)	-13.13226 (0.0000)	-15.47148* (0.0000) **	-22.40386 (0.0000)	-23.60082 (0.0000)	-22.15690 (0.0000)
KPSS***	0.34000	0.089873	0.143690	0.096993	0.345968	0.126374
Panel C: ARCH-LM tests: ARCH (1)- Optimal lag:1						
F-statistic	22.47599 (0.0000)	28.89527 (0.0000)	49.48604* (0.0000) **	24.93015 (0.0000)	20.98615 (0.0000)	11.01400 (0.0009)
Obs*R-squared	22.02457 (0.0000)	28.13774 (0.0000)	47.24053* (0.0000) **	24.37076 (0.0000)	20.59474 (0.0000)	10.91551 (0.0010)

*Denotes rejection of the null hypothesis at the 1% level.
**Denotes significance at 1% level.
*** Asymptotic critical values for the KPSS test are 0.739, 0.463 and 0.347 at the 1, 5 and 10% levels, respectively.
Source: *Research findings and http://tsetmc.ir/*

After evaluating the time series according to the main test, different types of GARCH models are estimated according to different values of p and q. Then, the optimal model is selected according to the minimum AIC, minimum SIC or minimum HQIC. The results show that EGARCH (2, 2) is the optimal model for all series. Therefore, EGARCH (2, 2) is used to estimate the conditional volatility of return in the parametric approach. Finally, the mean-EGARCH (2, 2) VaR portfolio selection model is ready to analyze on the basis of the knapsack problem.

Also, for the historical-VaR calculation, a confidence level (α) is considered 95%. Therefore, the mean-historical VaR portfolio selection model based on the knapsack problem is ready for analysis and comparison with the results of the mean-EGARCH (2, 2) VaR portfolio selection model.

Table reports the information of historical VaR and the conditional volatility of return of each stock. In addition, the upper bound, lower bound, return and price of each stock are reported in Table 2:

Table 2. Information of historical VaR, conditional volatility of return, price, return, lower and upper bound of each stock

Number of stocks	$\bar{P}_i$	$\bar{R}_i$ %	l_i	u_i	historical-VaR(95%)	$\sigma^2_{i,t}$ EGARCH(2,2)
S_1	5180.8	0.1946	2	15	4.7742	3.831068
S_2	14758	0.2493	3	20	4.8223	11.405114
S_3	9046.7	0.2796	5	23	4.86	12.576639
S_4	6237.3	0.3442	5	100	4.7892	6.061393
S_5	3095.9	0.1370	1	10	4.7525	44.883326
S_6	5022.3	0.2533	11	27	4.7801	7.4934116
S_7	8487.7	0.3157	10	45	4.7105	3.990445
S_8	6140.9	0.2467	7	30	4.7287	4.124969
S_9	8626.5	0.2658	3	40	4.696	20.785119
S_{10}	6013	0.2641	12	80	4.6683	2.136817
S_{11}	25795	0.2350	8	21	4.6578	9.284352
S_{12}	22281	0.1440	10	100	4.6007	4.849718
S_{13}	22759	0.1649	10	28	4.5214	4.144770
S_{14}	31900	0.2988	20	100	4.5370	0.256803
S_{15}	3699.9	0.1552	8	17	4.6695	16.890506
S_{16}	7623.8	0.2844	2	13	4.857	13.977827
S_{17}	18587	0.2408	1	12	4.4347	0.790044
S_{18}	17689	0.3246	9	26	4.8029	4.904840
S_{19}	36614	0.3611	4	19	4.6210	12.855087
S_{20}	19931	0.2419	5	18	4.8267	5.440757

Source: *Research findings and http://tsetmc.ir/*

Note that the lower and upper bounds of each share are randomly created and the budget is assumed to be 3000000IRR.

Therefore, the exact solutions of the mean-historical VaR portfolio optimization based on the knapsack problem are calculated using GAMS software. All results are presented in the Table 3:

Table 3. Exact solution of the mean-historical VaR portfolio optimization with different cardinality

	K=5	K=6	K=7	K=8	K=9
$E(R_P)$	15.747%	15.358%	14.766%	14.096%	13.304%
$VaR(R_P)$	42.538%	57.005%	75.489%	99.435%	129.384%
x_i	X_1=2; X_4=33; X_5=1; X_6=13; X_{16}=2	X_1=2; X_4=29; X_5=1; X_6=12; X_{15}=8; X_{16}=2	X_1=2; X_4=23; X_5=1; X_6=11; X_8=7; X_{15}=8; X_{16}=2	X_1=2; X_4=18; X_5=1; X_6=12; X_8=7; X_9=3; X_{15}=8; X_{16}=2;	X_1=2; X_4=15; X_5=1; X_6=12; X_8=7; X_9=3; X_{15}=8; X_{16}=2; X_{17}=1

Source: *Research findings*

In the following, the results of the exact solutions are compared with the NSGA-II solutions in the mean-historical VaR portfolio optimization problem. Hence, the Pareto optimal frontiers of the exact method and NSGA-II must be provided. Therefore, to find the exact Pareto sets in the mean-historical VaR portfolio optimization problem, $\varepsilon - constraint$ method is used t. In additions, the following is used to find the NSGA-II Pareto sets in the mean-historical VaR portfolio optimization problem.

In the study, the initial population is randomly generated. Generic encoding is provided in the form of a simple example of the representation in Figure 2:

	S 1	**S 2**	**S3**	**…**	**Sn-2**	**Sn-1**	**Sn**
$y_i = \{y_1, y_2, ..., y_n\} \in \{0,1\}$	1	0	1	…	0	1	0
$x_i = \{x_1, x_2, ..., x_n\} \in \text{int}$	5	3	10	…	6	12	9

Figure 2. Generic encoding.
***Source**: Research findings*

In Figure 2, the first row is a binary string that shows whether an asset is involved in a portfolio. The second row contains a string of integers that represents the number of each asset and is located between its upper and lower bounds.

In addition, a single point crossover is provided to produce a parent population. In this operator, a random position is selected along the string and the genes of the two parent chromosomes together change from the selected position to the end of the string (see Spears & De Jong, 1991). In addition, a swap mutation for permutation is considered to generate an offspring population. This operator chooses two genes from a chromosome randomly and then changes their positions (see Holland, 1992).

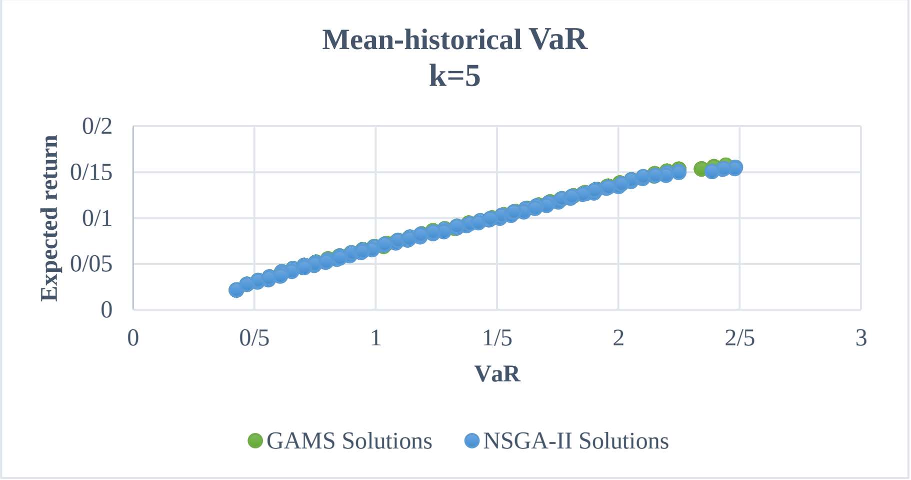
Next, the Taguchi method is used to select the appropriate parameters such as crossover rate (p_c) , mutation rate (p_m) , number of iterations and population size of the NSGA-II at three different levels. {0.7, 0.8, 0.9}, {0.05, 0.2, 0.3}, {50, 100, 500} and {100, 500, 1000} are employed for the crossover and mutation rates, population size and number of iterations, respectively. This method is used for both proposed models and both sets of objective functions. Therefore, the results of the proposed NSGA-II parameter adjustment for both models and both sets of objective functions based on the Taguchi method are reported in Table 4:

Table 4. Parameter setting of NSGA-II

	Crossover rate	Mutation rate	Population size	Number of iterations
Mean- historical VaR model	0.8	0.3	100	1000
Mean- EGARCH (2, 2) VaR model	0.9	0.2	100	500

***Source**: Research findings*

The GAMS Pareto optimal frontiers and the NSGA-II Pareto optimal frontiers of the mean-historical VaR portfolio optimization on the basis of the knapsack problem with cardinality k = 5, 6 are shown in Figure 3.



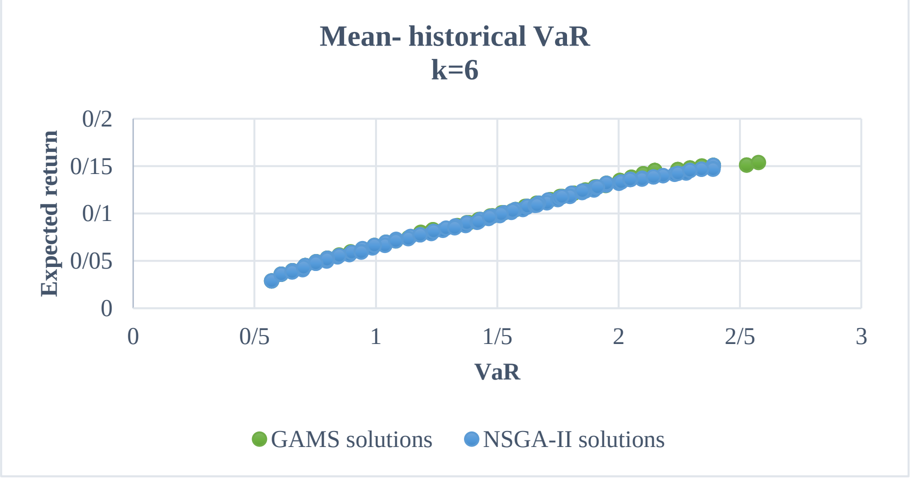
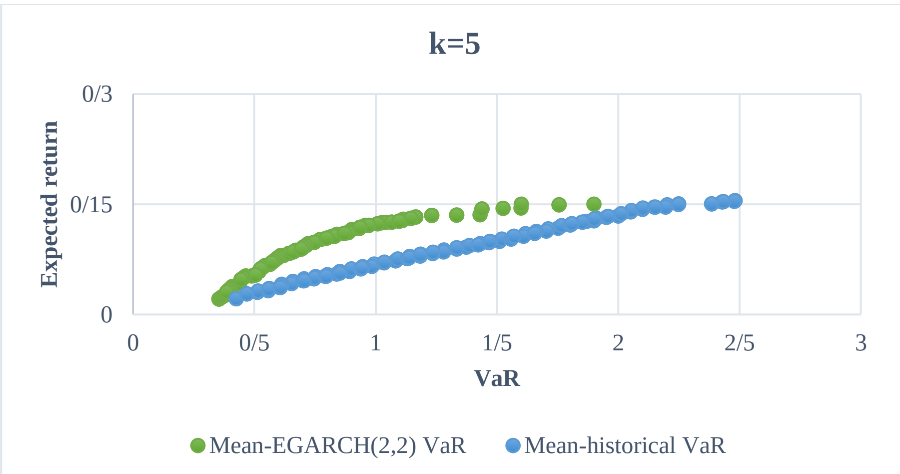


Figure 3. Exact and NSGA-II Pareto optimal frontiers of the mean- historical VaR portfolio optimization

***Source**: Research findings*

As shown in Figure 3, the NSGA-II Pareto optimal frontiers are very close to the exact Pareto optimal frontiers in the mean-historical VaR portfolio optimization problem. Thus, these results demonstrate the validity used by NSGA-II for the mean-historical VaR portfolio optimization problem.

Then, the NSGA-II Pareto optimal frontiers of the mean-EGARCH (2, 2) VaR portfolio optimization are calculated based on the knapsack problem. Finally, the NSGA-II Pareto optimal frontiers of two models are compared with each other in Figure 4.



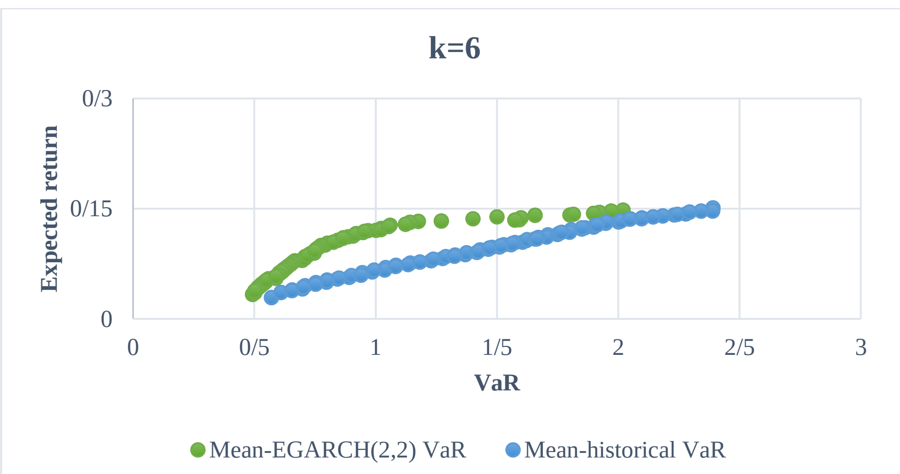


Figure 4. NSGA-II Pareto optimal frontiers of the mean- historical VaR portfolio optimization and the mean-EGARCH (2,2) VaR portfolio optimization

***Source**: Research findings*

According to Figure 4, the VaR in the mean- EGARCH (2, 2) VaR portfolio optimization problem is less than the mean-historical VaR portfolio optimization problem at a constant level of the return. In other words, the Pareto optimal frontiers of the mean-EGARCH (2, 2) VaR portfolio selection model are superior to the Pareto optimal frontiers of the mean- nonparametric VaR portfolio optimization problem. In other words, the Pareto optimal frontiers of the mean-EGARCH (2, 2) VaR portfolio selection model are superior to the Pareto optimal frontiers of the mean- nonparametric VaR portfolio optimization problem.

5. Conclusions

In this paper, we have focused on the problem of Markowitz asset allocation that the cost of selecting a particular asset is relatively high, for example, Berkshire Hathaway Inc. Therefore, we have introduced a multi objective portfolio selection model based on the improved knapsack problem, taking into account the cardinality, quantity and budget constraints. The first objective function maximizes the expected return of the portfolio and the second objective function minimizes the portfolio risk value (VaR) for risk assessment in the knapsack-based portfolio selection model. We have considered parametric (variance-covariance matrix) and nonparametric (historical) approaches to measure VaR. We have used the best of GARCH family models to estimate the conditional return fluctuations in the variance-covariance matrix, which is based on measuring and comparing different criteria in different types of GARCH family models.

Finally, MOEA has been planned for the solution. To show the practical application of the proposed model, a real portfolio of the Iran stock market has been solved.

By comparing the results of two sets of objective functions based on different approaches in risk measurement, a superior approach in risk assessment has been revealed for the knapsack- based portfolio selection model.

Therefore, the results of two sets of objective functions have been calculated based on the different approaches in risk assessment. The GAMS Pareto optimal frontiers and the NSGA-II Pareto optimal frontiers of the mean-historical VaR portfolio optimization have been compared with each other. The NSGA-II Pareto optimal frontiers are very close to the exact Pareto optimal frontiers in the mean-historical VaR portfolio optimization problem. Therefore, the results show that the validity of the applied NSGA-II for the mean-historical VaR portfolio optimization problem.

Then, the NSGA-II Pareto optimal frontiers of the mean-EGARCH (2,2) VaR and mean historical-VaR portfolio optimization have been compared with each other.

As a result,, Pareto optimal frontiers of the mean-EGARCH (2, 2) VaR portfolio selection model were superior to the Pareto optimal frontiers of the mean- nonparametric VaR portfolio selection model. These results show the importance of choosing the right method for measuring risk in portfolio optimization problems.

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Appendix

Appendix A. data. Related information can be found at https://u.pcloud.link/publink/show?code=XZ6PFMXZtNSfwgtbXepbYJkuwT7NiySgbHGy