Iranian Journal of Economic Studies, 10(1) 2021, 31-55

Iranian Journal of Economic Studies

Journal homepage: ijes.shirazu.ac.ir

Steady State Behavior of the Iranian Economy with Stochastic Energy Resources

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Article History

Received date: 03 October 2020
Revised date: 10 July 2021
Accepted date: 25 July 2021
Available online: 25 August 2021

JEL Classification

C13
C41
C61
O41

Keyword

Stochastic Growth Model
Exhaustible Energy Resources
Steady State Behavior
Iranian Economy

Abstract

The pertinent question is whether scarcity of non-renewable energy resources limits economic growth. Given that the earth's natural resources are limited; the answer appears to be yes. However, there are two reasons to reject this question. Technological advancements that conserved resources may be able to eliminate resource scarcity. Additionally, countries can import resources from other countries. This paper aims to develop an endogenous growth model with stochastic exhaustible energy resources and use it to explain the economy's steady state behavior. We consider the uncertainty associated with extractable energy resources and then develop a stochastic growth model on this basis. Additionally, we solve this model analytically using the Stochastic Hamilton-Jacobin-Bellman method (SHJB method). Finally, for the Iranian economy, we apply the analytical solution. The primary findings indicate that as natural resource extraction becomes even more uncertain, the rate of economic growth slows, which results in a subsequent decline in the rate of resource extraction. Furthermore, we observe that the variance in energy extraction in the Iranian economy is approximately 0.22. Under these conditions of uncertainty, the optimal economic growth rate in a steady state will be 7.1 percent with an extraction rate of 1.1 percent.

Highlights

- An endogenous growth model with stochastic exhaustible energy resources was developed.
- we solve this model analytically using the Stochastic Hamilton-Jacobin-Bellman method (SHJB method).
- the model calibrated for Iranian economy, optimal economic growth rate in a steady state and extraction rate were calculated.

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DOI: 10.22099/ijes.2021.38645.1715

1. Introduction

Numerous studies have been conducted in an attempt to construct a theory of economic growth. The majority of these studies were conducted based on specific assumptions. However, it is clear that most resources, such as energy, are stochastic in some way. As a result, it is necessary to augment existing growth models with stochastic growth models to explain economic behavior better.

Countries with more natural resources have a greater chance of achieving higher growth and improving household welfare, as they can use natural resources such as petroleum and natural gas to generate output and physical and human capital. Excessive resource use is impossible, at least in the long run, because the earth's natural resources are finite. Thus, it is easy to imagine that we will exhaust all of the country's non-renewable resources at a point in the future. Without the ability to use resources in production, economic growth will slow and eventually reach a standstill. This is the most heinous scenario imaginable.

(Solow, 2009) demonstrated that resource depletion is a minor, if not non-existent, concern. Taking note that if natural resources are exhausted, their prices must increase rather than decrease. Indeed, (Aguilera & Ripple, 2012) estimate that oil and gas are more abundant than previously believed, at least in Europe. Additionally, according to (Ostadzad & Hadian, 2017), the trend of extractable energy natural resources is increasing for the Iranian economy.

There are two reasons for the suspension of natural resource scarcity. Specifically, (Dasgupta & Heal, 1974) argue that technological advancement has rendered previously necessary exhaustible resources obsolete. Second, countries can structure for the absence of any resources by importing them elsewhere (Weil, 2014). In reality, if a country with limited natural resources requires petroleum for manufacturing purposes, it can import it from countries with abundant energy resources. These two assessments do not make a strong case for the indefinite avoidance of finite natural resources. We will concentrate on Iran as a country endowed with abundant natural resources (Dasgupta & Heal, 1974). Technological progress that substitutes for resources may not occur, which is why Dasgupta uses a stochastic model with an indeterminate discovery.

Several studies, including (Velenturf & Purnell, 2017), (Barbier, 2021), (Polasky et al., 2019), and (Couix, 2019) have discovered the implications of resource scarcity for economic growth.

Optimistic economists believe that technological advances lead to discovering new reserves and could likely compensate for the increase in demand for non-renewable resources. In contrast, pessimistic economists believe that these effects are not sufficiently reliable (Hettich, 2000). The trend of oil and gas resources shows that pessimistic views are misleading. It can be agreed that, rather than depleting oil and gas reserves, these reserves have increased over the last decades (McGlade, 2014).

(Cheviakov & Hartwick, 2009) expanded the Solow model by including exhaustible resources. The demonstration that the higher rate of physical capital depreciation terminates the economy can be avoided by robust technological improvement.

(Aghion, Blundell, Griffith, Howitt & Prantl, 2009) demonstrated that the creative destruction (or Schumpeterian) growth model could be maintained in the presence of exhaustible resources.

Romer describes how technological progress would make it possible to sustain growth within the framework of neoclassical growth models, despite the presence of resource depletion and land scarcity (Jones, 2019).

None of the prior studies examined the uncertainty inherent in resource dynamics. However, it is acknowledged that resource dynamics are stochastic in nature.

In continuation of (Dai, Kou & Qin, 2019), we present a stochastic model in which a geometric Brownian motion stochastic process drives resource dynamics. Additionally, we take into account zero degrees of openness. This supposition enables the analytical solution to be deduced. In contrast to (Smith, 2007), which only solves a few of this class of stochastic growth models in closed form, we solve the model analytically, resulting in clear and transparent expressions that can be used to answer the research questions. Then we use those to investigate how increased uncertainty affects economic growth and agent welfare. The findings indicate that increased uncertainty reduces economic growth and welfare.

To summarize, this paper will use analytical techniques to examine the implications of natural resource scarcity and associated uncertainty for growth and welfare. The following is the organization of the paper. Sections 2 and 3 establish the context for the literature review, while the model defines and discusses the model's implications. Sections 5 and 6 contain solutions and concluding remarks.

2. Literature Review

This section summarizes research on the stochastic growth model. Additionally, the growth model based on natural resources was deliberate. (Tilton, 1996) examined both optimistic and pessimistic perspectives on long-term economic growth in light of resource depletion, uncertainty, and technological development in his study. (Pasqual & Souto, 2003) examined the issue of long-term economic growth in light of resource uncertainty. They established that intergenerational resource distribution ensures sustained economic growth.

Also (Papyrakis & Gerlagh, 2004) addressed the long-term economic growth trajectory concerning the limitations of intergenerational and non-renewable sources with certainty. They demonstrated that economic convergence is dependent on the stock of primary sources.

(Martinet & Doyen, 2007) used optimal control to analyze long-term economic growth in the presence of non-renewable resources, technology, and a lack of other uncertainties in their article. Another study examined the potential for long-term economic growth in light of non-renewable resource depletion, endogenous extraction rates, and a dynamic population (da Silva, 2008).

(Schilling & Chiang, 2011) a theoretical article examined optimism and pessimism regarding long-term economic growth, considering externalities, resource depletion uncertainties, technological advancement, and resource substitution.

(Mitra & Roy, 2017b) define a new set of conditions for a policy function that satisfies the Ramsey–Euler equation to be optimal in a single sector stochastic optimal growth model. This paper concludes that an interior Ramsey–Euler policy function is optimal if it is continuous or if both consumption and investment are non-decreasing in output. Under these conditions, the policy's stochastic paths must satisfy the transversality constraint. Applying this study implies that future research will not need to verify the transversality condition when determining the optimality of a policy function.

(Mitra & Roy, 2017a), a condition for output and consumption to be strictly positive with probability one, in the long run, was derived in a general model of stochastic optimal economic growth. The condition establishes that output in the distribution has converged to a unique positive stochastic steady state. This study condition explicitly involves a near-zero level of relative risk aversion and is weaker than existing literature conditions. (Mitra & Roy, 2017a) analyzes productivity and relative risk aversion in a framework where productivity and relative risk aversion are constrained, and productivity shocks are, i.i.d. in time.

Additionally, it demonstrated the tightness of the condition for the family of constant relative risk aversion (CRRA) utility functions, in which a strict violation of their condition implies almost certain global convergence to zero. For this family of utility functions, output and consumption may almost certainly converge to zero when risk aversion at zero is near 1. Convergence to a positive steady state is ensured when risk aversion is either sufficiently small or sufficiently large. It is neither necessary nor sufficient for global convergence to a positive steady state if expected productivity at zero is greater than the discount rate.

(Rubini & Moro, 2019) propose a controllable algorithm for solving structural change stochastic growth models. Structural change, in general, indicates an unbalanced growth path. When uncertainty is introduced, this property precludes using local solution techniques and necessitates using global methods. The algorithm was applied to a stochastic version of a three-sector structural transformation growth model with Stone-Geary preferences using the Parameterized Expectations Approximation. The calibrated explanation was used to demonstrate a tight connection between the economy's long- and short-run properties in this class of models. This tension arises from the non-homothetic components of the various forms of consumption required to accommodate long-run structural change. However, it implies high volatility of services and low volatility of manufacturing and agriculture in the short run.

The following question is posed in (Tsuboi, 2019): "Will natural resource scarcity constrain economic growth?" Two scenarios were developed to address this question "initially, resource-conserving technological progress has the potential to eliminate resource scarcity. Second, countries may import resources from other countries '".

Further details can be found in the literature by (Talari, Shafie-Khah, Osório, Aghaei & Catalão, 2018), (De La Fuente-Mella, Vallina-Hernandez & Fuentes-Solís, 2019), (Tsuboi, 2019), (Di Somma, Graditi, Heydarian-Forushani, Shafie-Khah & Siano, 2018).

Although several studies examined the effects of non-renewable primary energy resources on economic growth as non-random, the stochastic growth model does not consider the effect of extractable random energy resources on economic growth. The random level for extractable resources is considered in this study, and a stochastic growth model is developed. We solve our model using the analytic method (Stochastic Hamilton Jacobin Bellman Method). Then, we calibrate the solved model for the Iranian economy to analyze the economy's steady state behavior.

3. The Model

The model presented in this article is a modified version of Ramsey's classic growth model. We assume that energy is a stochastic variable that plays a significant role in the production sector. Imposing this component on the Ramsey traditional growth model justifies an economy's steady state behavior.

The study makes the assumption of a closed economy with three sectors. The model's fundamental premise is that there will be no government and that the government will serve solely as a social planner.

The following sections survey economic sectors (households, businesses, and the energy sector). In the next section, a model for Iran's economy is developed that accounts for the country's fluctuating energy resources.

3.1 Households

This model's economy is composed of a large number of identical households that can exist indefinitely. Through the following welfare function, each household attempts to maximize its total utility.

$$W(.) = Max \int_0^\infty u(c)e^{-\rho t}dt \tag{1}$$

Where $u(c)$ is the instantaneous utility function and $u_c = \frac{du}{dc} > 0$. According to (Besov, 2014) (Chakravarty & Manne, 1968) and (Aseev & Kryazhimskii, 2007) the functional form of instantaneous utility is given in equation (2).

$$U(C_t) = \frac{C^{1-\varsigma}}{1-\varsigma} \tag{2}$$

$\frac{1}{\varsigma} > 0$ Shows private consumption's intertemporal elasticity of substitution between two consecutive points of time. Also $\rho > 0$ is discount rate.

Furthermore, the household's budget constraint is,

$$dK = (Y_t - C_t - \delta K)dt \Rightarrow \dot{K} = Y_t - C_t - \delta K \quad (3)$$

$\dot{K}$ and K represent investment and capital stock, respectively, δ is the depreciation rate of capital stock and C_t is private-sector consumption. Households attempt to maximize equation (1) according to budget constraints (eq.3). The following sections discuss the sectors of firms.

3.2 Firms

Numerous similar firms produce a variety of different goods. The firms in this study are classified into two categories: those producing finished goods and those that produce energy. Each firm generates output through the use of capital and labor.

Inputs used by finished goods producers are capital, energy, and labor. The production function has been assumed according to a Cobb-Douglas form in equation (4) (Cheng & Han, 2014), (Matsumoto & Szidarovszky, 2021).

$$Y_t = A(\eta K_t)^{\alpha}(\theta H_t)^{\beta}(E_t)^{\chi} \quad (4)$$

In equation (4) η is the percentage of the capital used in the production of finished goods and $1 - \eta$ is the percent of capital in the energy sector. Also θ is the percentage of labor used to produce finished goods and $1 - \theta$ is the percent for energy.

Where $E, \alpha, \chi, \beta, \eta, \theta$ is energy consumption in the finished goods sector, the elasticity of production concerning capital, the elasticity of production concerning energy consumption, the elasticity of production concerning labor, the share of capital in the finished goods sector, and the labor share in the finished goods sector, respectively.

Also, we assume energy production is given by equation (5). It is assumed that in the energy production sector, three inputs have been used (capital, labor, and primary energy (rR_t)).

$$E_t = B[(1-\eta)K_t]^{\gamma}[(1-\theta)H_t]^{\lambda}[rR_t]^{\kappa} \quad (5)$$

Where $\gamma, \kappa, \lambda, 1-\eta, 1-\theta$ is the elasticity of energy production concerning capital, the elasticity of energy production concerning extracted resources, the elasticity of energy production concerning labor, the share of capital in the energy generation sector, and the share of the labor force in the energy generation sector, respectively.

In relation (5), rR_t is the extraction of primary energy resources and is assumed to be the final energy used in producing the finished goods. Therefore, r is the extraction rate of oil and gas. Since the level of resources is limited and changes with extraction, the equation of motion is surveyed for extractable energy resources.

3.3 The Equation of Motion for Extractable Energy Resources

Pindayk (1980) classifies exploration activities into two categories. First, the discovery of new information about the distribution of reserves, mainly when this distribution is unknown. In this instance, exploration functions similarly to a Bayesian Learning Process (Clark & Mangel, 1986). Second, exploring new areas will discover new reserves, allowing for expanding available resources. As a result, each exploration of existing reserves may increase rather than decrease, according to a random process. This is especially true if we conduct a long-term analysis of the data to apply macroeconomic models.

Similar to Reed (1979) we introduce a natural growth rule as a function of resource in discrete-time,

$$Q_{t+1} = Z_t \times R(Q_t) \quad (6)$$

Where $R(.)$ is a reproduction of expected resources and $Z_t$is defined as the average of all random variables that are distributed independently and identically and also exhibit random shocks. These random shocks can affect the growth of the resource stock and the expected level of resources in each period. The level of resources becomes evident after realizationZ_t.

However, the future evolution of stock processes cannot be predicted. In general, the concept of a steady state should be replaced by the concept of steady state distribution in this class of stochastic models. However, if the random shock realizations are observed before making extraction decisions, the optimal policy maintains a constant escapement, i.e., the optimal escapement distribution degenerates to a constant (Reed, 1979). When additional sources of uncertainty are introduced (e.g., errors in the measurement of current stocks), the constant escapement rule breaks down. (Leizarowitz & Tsur, 2012) used Ito's stochastic calculus to formulate the resource management problem in continuous time with stochastic stock evolution. The stock's evolution is determined by a diffusion process that follows the stochastic differential equation (Eq. 7).

$$dQ = [R(Q) - q]dt + \sigma(Q)dz \quad (7)$$

Where in this equation, Z is a standard Wiener process and $\sigma^2(.)$ is the corresponding variance. Specifying$\sigma(Q) = \sigma Q$, with a σ constant, gives rise to a geometric Brownian motion and greatly facilitates the analysis. By using the expected cumulative net benefit as the optimization objective, stochastic dynamic programming can derive the optimal extraction rule and associated steady state distribution. Again, the prudence implications of this type of uncertainty are ambiguous and dependent on the recharge and benefits functions' properties. See (Kitabatake, 1989) and (Figuières & Tidball, 2012) for examples in which the optimal exploitation rule $q(Q)$ increases, remain unchanged, or decreases as the variance parameter σ increases. Other examples of resource management under stochastic stock dynamics include (Plourde & Yeung, 1989), (Knapp & Olson, 1995), (Wirl, 2006a), and (Wirl, 2006b).

In this study, changes in the level of extractable energy resources (dR) is considered in equation (8). It has two components, random section $(R_t\sigma_R dz)$ and non-random component $(-rR_t dt)$.

$$dR = -rR_t dt + R_t\sigma_R dz \quad (8)$$

The augmented Ramsey growth model with stochastic resources (oil and gas primary resources) has been investigated. Thus, concerning sections 2.1-2.3, the following section summarizes the problem (which a social planner must resolve).

3.4 Augmented Stochastic Growth Model

Assume a benevolent social planner selects consumption choices over time, intending to maximize the representative individual's welfare. Given that individuals are identical, we assume that the social planner assigns each agent the same level of consumption. Consequently, the instantaneous utility of each individual is given by$U(C_t) = \frac{C^{1-\varsigma}}{1-\varsigma}$. The social planner's optimization problem is then to choose c (t) to maximize $U(.) = MaxE \int_0^\infty u(c)e^{-\rho t}dt$, subject to the economy's resource constraint (equations 3 and 8), and the initial condition k (0) = k0. Thus, the stochastic optimization problem that social planners face can be depicted as follows:

$$W(.) = MaxE\left(\int_0^\infty \left(\frac{{C_t}^{1-\varsigma}}{1-\varsigma}\right)e^{-\rho t}dt\right)$$

$$s.t$$

$$dK = \left(A(\eta K_t)^\alpha (\theta H_t)^\beta (E_t)^\chi - C_t - \delta K_t\right)dt$$

$$dR = -rR_t dt + R_t \sigma_R dz$$

$$dH = nH_t dt$$

$$E_t = B[(1-\eta)K_t]^\gamma [(1-\theta)H_t]^\lambda [rR_t]^\kappa \quad (9)$$

Because the social planner must solve an optimal control problem (problem 9), we have included an explanation of the stochastic optimal control mathematical model in the following section. Following that, stochastic optimization (problem 9) is solved analytically with reference to the standard Mathematical stochastic optimal control model.

4. Methodology (Mathematical Model of the Stochastic Optimal Control Problem)

Optimal control is a mathematical branch of the control theory used in many economics fields, such as growth models and financial models. It aims to discover a control law for a controlled stochastic or ordinary dynamical system while minimizing or maximizing some utility function.

Deterministic optimal control has been linked to industrial applications since its beginning in the 1950s, starting with aerospace (problem of an optimal trajectory of an airplane). Stochastic optimal control appeared later on in the 70s in the financial sector. (Merton, 1975) studied the stock portfolio optimization. (Black & Scholes, 2019) presented the notion of a financial model. The optimal control problems developed by Bellman and optimal control problems can be solved by the dynamic programming method.

Bellman states that the value function associated with the optimal control problem satisfies particular equality called a dynamic programming principle (DPP). (Bellman, 1952; Bellman & Kalaba, 1965; Bellman, 1962). Classical control theory evolved into the new era of modern control theory, emphasizing the importance of controllers having an optimality property. The development of

Pontryagin's maximum principle and Bellman's dynamic programming initiated this new discipline in the late 1950s and early 1960s (Derakhshan, 2015).

The purpose of the remainder of this section is to provide a mathematical model for stochastic control problems. Indeed, we propose to use mathematics to explain a stochastic control problem, and we coin the term Stochastic Controlled Dynamics[1].

Any stochastic control dynamic problem in economics has two segments.

1- Stochastic Controlled Dynamics equation

2- Payoff, in that the growth model payoff optimization equals welfare optimization.

First, explained the Stochastic Controlled Dynamics equations are explained. Assuming that f depends upon some "control" so that $f: R^n \times A \to R^n$ where $A \subseteq R^m$. We call a function $a: [t_1, T] \to A$ control, where t1 and T are initial and terminal time, respectively. Concerning each control function, we deliberate System Dynamic Equation (SDE)

$$\begin{cases} dx(t) = f(x(t), a(t))dt + \sigma dB_t t_1 \leq t \leq T \\ x(t_1) = x_1 \end{cases} \tag{10}$$

Where B_t is Brownian motion, and σ is volatility. We denote the trajectory x(.) as the corresponding response of the system to control a(.). We also assume a(.) is measurable and note the collection of all acceptable controls, where $a(t) = \{a_1(t), a_2(t), \dots, a_m(t)\}$

The task will be to determine the "best" control for the system. So, we must describe a standard payoff (or welfare in the growth model). Finally, the payoff is defined as functional, and because the system is stochastic, it is defined as expectation, so

$$W\big(a(.)\big) = max\, E\left(\int_{t_1}^{T} u(x(t), a(t))dt + g(x(T))\right) \tag{11}$$

Where x(.) satisfies SDE (10) for the control a(.).

Here $u: R^n \times A \to R$ and $g: R^n \to R$ are given, and we call u the running payoff (instantaneous utility) and g the terminal payoff (terminal utility). The terminal time T > 0 is given. In welfare function (11) with regards to SDE (10), The primary purpose is to find a control a∗(.), which maximizes the welfare. In other words, we want to achieve $W\big(a^*(.)\big) > W\big(a(.)\big)$, for all controls a(.) ∈ A. Such a control a∗ (.) is called optimal control.

Specific problems in mathematics are difficult or impossible to solve directly. The novel approach is to reclassify the problem as part of a larger family of similar problems through revision. The Hamilton-Jacobi-Bellman equation is the primary tool we use in dynamic programming algorithms.

Assuming that the value function V is a function of variables (x,s), $x \in R^n$, 0 ≤ s ≤ T. Then V solves the following semilinear parabolic PDE

[1] It should be declared that this would not be a complete explanation; because we do not need to enter the complicated deliberations of mathematical analysis. We only provide a simple definition which satisfies our purposes in this paper.

$$\frac{\partial V(x,s)}{\partial s}+\frac{1}{2}\sigma^2 \Delta V(x,s)+\max_{a\in A}\{f(x,a).\nabla_x V(x,s)+u(x,a)\}=0 \quad (12)$$

Where $\nabla_x V(x,s)=\left(\frac{\partial V(x,s)}{\partial x_1},\frac{\partial V(x,s)}{\partial x_2},\dots,\frac{\partial V(x,s)}{\partial x_n}\right)$ is the gradient in terms of x. The PDE (12) is the stochastic Hamilton-Jacobi-Bellman (HJB) equation.

The PDE could solve as analytical or numerical. In this paper, we solve the stochastic Hamilton-Jacobi-Bellman (HJB) equation by the analytical method. Solving the SHJB equation has been considered in numerous studies, including (Kolokoltsov, 1996), (Kosygina, Rezakhanlou & Varadhan, 2006), (Bayraktar & Sirbu, 2013), (Qiu, 2018), (Wu & Yu, 2008) and (Fuhrman, Masiero & Tessitore, 2010).

The following section will discuss how to solve the optimal control problem (9) concerning PDEs (12).

5. Model Solving and Analyzing

In the preceding section, we discussed models with a limited time horizon and optimal control. However, the time horizon of the augmented stochastic model used in this study is unbounded. To convert a stochastic dynamic model with a finite time horizon to an infinite time horizon, see (Aliyu, 2018), (Krener, 2019) and (Satoh, Kappen & Saeki, 2016).

Concerning section 3, the model has three equations of motion (equation of motion for capital (K), labor (H), and stochastic extractable energy resources (R)). Therefore, the state variables are K, R, and H. The state variables vector is defined in section 3 by x vector. So$\vec{x}(t)=[K_t,R_t,H_t]$.

Control variables are consumption (C_t), the rate of energy resources extraction (r), the percentage of the labor force in the finished goods-producing sector (θ), and the percentage of capital in the finished goods-producing sector (η). Hence, reference to the control variable defined in section 3, the vector of control variables is$\vec{a}(t)=[C_t,r,\theta,\eta]$.

This section solves the stochastic intertemporal utility optimization problem using capital, labor, and random extractable resources through motion equations (eq. 13).

$$W(.)=MaxE\left(\int_0^\infty\left(\frac{{C_t}^{1-\varsigma}}{1-\varsigma}\right)e^{-\rho t}dt\right)$$
$$s.t$$
$$dK=\left(AB^\chi r^{\kappa\chi}\eta^\alpha(1-\eta)^{\gamma\chi}\theta^\beta(1-\theta)^{\lambda\chi}{K_t}^{\alpha+\gamma\chi}{H_t}^{\beta+\lambda\chi}{R_t}^{\kappa\chi}-C_t-\delta K_t\right)dt$$
$$dR=-rR_t dt+R_t\sigma_R dz$$
$$dH=nH_t dt \quad (13)^2$$

For this purpose, we use the stochastic Jacobins Hamilton, Bellman equation (SHJB) as follows concerning control variables ($\vec{a}(t)=[C_t,r,\theta,\eta]$) and state variables ($\vec{x}(t)=[K_t,R_t,H_t]$) vectors.

[2] In optimal control problem (eq.9), energy production function have replaced in capital equation of motion.

$$\rho V(\vec{x}) = \max_{\vec{a}} \left\{ u(\vec{x},\vec{a}) + f(\vec{x},\vec{a}).\nabla_x V(x) + \frac{1}{2} tr(\Delta V(x) \times \sigma^2) \right\} \quad (14)^3$$

In equation (14), $\vec{\nabla} V$ and ΔV are the gradient vector and the Hessian matrix of V, respectively. $f(\vec{x},\vec{a})$ and $\vec{\sigma}$ are a deterministic and random component of equations of motions. The equations of motion in (13) has been denoted in a vector format (eq.15) (for simplicity, we replaced $Y_t = AB^{\chi} r^{\kappa\chi} \eta^{\alpha} (1-\eta)^{\gamma\chi} \theta^{\beta} (1-\theta)^{\lambda\chi} K_t{}^{\alpha+\gamma\chi} H_t{}^{\beta+\lambda\chi} R_t{}^{\kappa\chi}$ in (9))

$$\begin{bmatrix} dK_t \\ dR_t \\ dH_t \end{bmatrix} = \begin{bmatrix} Y_t - C_t - \delta K_t \\ -rR_t \\ nH_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ R_t \sigma_R \\ 0 \end{bmatrix} dB_t \quad (15)$$

Thus, according to the standard format (eq.10), $f(\vec{x},\vec{a})$ and $\vec{\sigma}$ can be defined as follows.

$$f(\vec{x},\vec{a}) = \begin{bmatrix} Y_t - C_t - \delta K_t \\ -rR_t \\ nH_t \end{bmatrix}, \vec{\sigma} = \begin{bmatrix} 0 \\ R_t \sigma_R \\ 0 \end{bmatrix} \quad (16)$$

Also, gradient vector and the Hessian matrix of $V(x,s)$ are $\vec{\nabla}_x V(x,s) = \left(\frac{\partial V(x)}{\partial K}, \frac{\partial V(x)}{\partial R}, \frac{\partial V(x)}{\partial H} \right) = [V_K \quad V_R \quad V_H]$ and $\Delta V(x,s) = \begin{bmatrix} V_{KK} & V_{KR} & V_{KH} \\ V_{RK} & V_{RR} & V_{RH} \\ V_{HK} & V_{HR} & V_{HH} \end{bmatrix}$, respectively. In the gradient vector and the Hessian matrix, V_X is the first derivative of V, and V_{XY} is the second derivative of V, where the set of X and Y are a set of state variables ($X = Y = \{K, R, H\}$).

The covariance matrix of the random component of equations is a motion

$$\sigma^2 = \sigma\sigma' = \begin{bmatrix} 0 \\ R_t \sigma_R \\ 0 \end{bmatrix} \begin{bmatrix} 0 & R_t \sigma_R & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (R_t \sigma_R)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

given by. Concerning the modeling section in this study, we considered only the energy resources to be a random variable, hence in a variance-covariance matrix, the only variance of extractable resources (R_t) is non-zero. All other components of this matrix must be zero.

Trace of multiplication of the Hessian matrix of V and variance-covariance

$$\frac{1}{2} tr\left(\Delta V \times \sigma^2 \right) = \frac{1}{2} V_{RR} \left(R_t \sigma_R \right)^2$$

matrix has to be equal to.

Also, the multiplication of the gradient vector of V and non-random component vector is shown in equations (9a).

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[3] Reference to (Achdou, Han, Lasry, Lions, & Moll, 2014), (Achdou, Buera, Lasry, Lions, & Moll, 2014) and (Nuño & Moll, 2018) SHJB defined in eq.12 can be written in this format.

$$f(\vec{x},\vec{a}).\nabla_x V(x) = (Y_t - C_t - \delta K_t)V_K - -rR_tV_R + nH_tV_H \quad (17)$$

Concerning calculating all components of SHJB equation (14), we have:

$$\rho V = \underset{C,r,\eta,\theta}{Max}\, u(C_t) + V_K(Y_t - C_t - \delta K_t) + V_R(-rR_t) + V_H(nH_t) + \frac{1}{2}V_{RR}(R_t\sigma_R)^2 \quad (18)$$

In equation (18), the derivative of SHJB relative to control variables should be zero. This means that ($\frac{d(\rho V)}{dC}=0$, $\frac{d(\rho V)}{dr}=0$, $\frac{d(\rho V)}{d\eta}=0$, $\frac{d(\rho V)}{d\theta}=0$). First-order conditions (FOC) results are shown in (19).

$$\begin{cases} \frac{\partial\rho V(x)}{\partial C_t} = 0 \xrightarrow{eq.(18)} u'(C_t) = V_K & \frac{\partial\rho V(x)}{\partial r} = 0 \xrightarrow{eq.(18)} V_K\frac{\partial Y_t}{\partial r} = V_R R \\ \frac{\partial\rho V(x)}{\partial\eta} = 0 \xrightarrow{eq.(18)} \frac{\partial Y_t}{\partial\eta} = 0 & \frac{\partial\rho V(x)}{\partial\theta} = 0 \xrightarrow{eq.(18)} \frac{\partial Y_t}{\partial\theta} = 0 \end{cases} \quad (19)$$

From (19), the optimal share of capital (η^*) and labor (θ^*) in the finished goods production sector are:

$$\eta^* = \frac{\alpha}{\alpha+\chi\gamma} \quad \theta^* = \frac{\beta}{\beta+\lambda\chi} \quad (20)$$

These equations confirm that, if the elasticity of inputs in the production of finished goods (elasticity of production relative to capital (α) and labor (β)) increase, the share of these factors in the production of finished goods will also increase (this means that increasing the efficiency of each production factor will increase the optimal share of that input in the finished goods production sector). Furthermore, regarding (20), if the elasticity of finished goods production relative to energy consumption increases ($\chi \uparrow$), the optimum share of capital and labor in the production of the finished goods must decrease ($\eta^* \downarrow, \quad \theta^* \downarrow$). This means that if the efficiency of final energy use in finished goods production increases, the share of labor and capital in the energy sector must also increase$(1-\eta^*)\uparrow, \quad (1-\theta^*)\uparrow$.

According to (Rathnayaka, Jianguo & Seneviratna, 2014), (Steele, 2012), (Cohen & Elliott, 2015), (Wang & Li, 2020), and (Brémaud, 2020), a vector form of Ito's Lemma for the finished goods production function is:

$$dY_t(\vec{x},\vec{a}) = \left[\overrightarrow{\nabla_x}Y_t.\vec{f} + \frac{1}{2}tr(\Delta_x Y_t.\vec{\sigma})\right]dt + \left[\overrightarrow{\nabla_x}Y_t.\vec{\sigma}\right]dB_t \quad (21)$$

Components of relation (21) must be calculated to determine the economic growth rate in a steady state. In this equation$\overrightarrow{\nabla_x}Y_t$ $\Delta_x Y_t$ are the gradient vectors of the production function and the Hessian matrix relative to the state variables. We will have a random differential equation of production (eq. 22) with the first and second partial derivative of the function and replacement of the production of finished goods (21).

$$dY_t = \left[Y_K(Y_t - C_t - \delta K_t) + Y_R(-rR_t) + Y_H(nH_t) + \frac{1}{2}Y_{RR}(R_t\sigma_R)^2\right]dt + [Y_R R_t\sigma_R]dB_t \quad (22)$$

In (22) $Y_K, Y_R, Y_H$are partial derivatives of the production of the finished goods relative to capital, natural resources, and labor, respectively ($Y_K =$

$\frac{\partial Y_t}{\partial K_t}, Y_R = \frac{\partial Y_t}{\partial R_t}, Y_H = \frac{\partial Y_t}{\partial H_t}$). Also $Y_{RR}$is the second derivative of output in regards to natural energy resources ($Y_{RR} = \frac{\partial^2 Y}{\partial R^2} = \chi\kappa(\chi\kappa - 1)\frac{Y_t}{{R_t}^2}$).

With regards to equations of motion of the capital and level of extractable energy resources ($\dot{K} = Y_t - C_t - \delta K$, $dR = -rR_t dt + R_t \sigma_R dz$) and dividing (22) to differential of time (dt), equation (22) can be simplified as:

$$\frac{dY_t}{dt} = Y_K \dot{K}_t + Y_H(nH_t) + \frac{1}{2}Y_{RR}(R_t\sigma_R)^2 + Y_R\frac{dR_t}{dt} \tag{23}$$

Replacing the production function derivatives in equation (23) and dividing this equation to (Y_t):

$$\frac{1}{Y_t}\frac{dY_t}{dt} = (\alpha + \chi\gamma)\frac{\dot{K}_t}{K_t} + (\beta + \chi\lambda)n + \frac{1}{2}\chi\kappa(\chi\kappa - 1){\sigma_R}^2 + \chi\kappa\frac{1}{R_t}\frac{dR_t}{dt} \tag{24}$$

Economy growth rate ($g_y = \frac{1}{Y}\frac{dY}{dt}$), the growth rate of capital ($g_k = \frac{\dot{K}}{K} = \frac{1}{K}\frac{dK}{dt}$) , and the growth rate of extractable resources ($g_R = \frac{dR}{dt}\frac{1}{R}$) were assumed and placed in (24). The economic growth rate is calculated concerning capital growth rate and the level of extractable resources.

$$g_y = (\beta + \chi\lambda)n + \frac{1}{2}\chi\kappa(\chi\kappa - 1){\sigma_R}^2 + (\alpha + \chi\gamma)g_k + (\chi\kappa)g_R \tag{25}$$

Although this equation for economic growth is insufficient for calibration, we can conduct some theoretical analysis. By analyzing this equation:

a) Increasing population growth rate ($n \uparrow$) will increase economic growth rate ($g_y \uparrow$).

b) Improvement in input factor's (labor, energy, and capital) efficiency $(\alpha, \beta, \chi) \uparrow$ increases the growth rate of the economy ($g_y \uparrow$), as a result.

Regarding (19) and the derivate of the utility function ($U'(C_t) = \frac{dU}{dC} = C^{-\varsigma}$) will result in $C^{-\varsigma} = V_K$. This relation shows that the marginal utility of consumption equal to the marginal value of capital.

Regarding (19) ($V_K\frac{\partial Y_t}{\partial r} = V_R R$), simultaneous utility function ($u(C_t) = \frac{C^{1-\varsigma}}{1-\varsigma}$) and some simplification will give:$\rho\Phi {K_t}^{1-\alpha-\chi\gamma} = \frac{{C_t}^{1-\varsigma}}{1-\varsigma}$. This relation shows the consumption function based on the capital. In terms of this relationship, countries with more capital consume more. On the other hand, investing now will result in future capital growth. Increased consumption in the future will result from increased capital.

By calculating the logarithm and then the differential of this relationship, the growth rate of capital can be calculated as (g_K) based on the growth rate of consumption (g_c) ($g_c = \frac{1-\alpha-\chi\gamma}{1-\varsigma}g_K$). Therefore, the consumption growth rate can be equal, equal, or less than the capital growth rate. Also, by solving equation systems (19), the growth rate of extractable energy resources can be calculated using eq. (26)

$$g_R = \frac{2\rho + (1-\chi\kappa)(2r - \sigma^2\chi\kappa) + 2r}{2\chi\kappa} \tag{26}$$

Relation (26) shows the growth rate of extractable resources (g_R) in a steady state based on the extract variance (σ^2) and a fixed rate of extraction (r).

We assume that the level of extractable resource growth rate fluctuations to be neutralized in a steady state. So $dR = -rR_t dt + R_t \sigma_R dz \xrightarrow{dz=0} \frac{1}{R_t}\frac{dR}{dt} = g_R = -r$. Placement of this result in equation (26) and the optimum extraction rate (non-random part of the equation of motion of extractable resource) will be calculated as the elimination of fluctuation in a steady state.

$$r^* = \frac{(1-\chi\kappa)(\sigma^2\chi\kappa) - 2\rho}{4} \tag{27}$$

According to equation (27) energy resources extraction rate will increase ($r^* \uparrow$) when volatility increases ($\sigma^2 \uparrow$) because $(1-\chi\kappa)$ is positive. This increase is because when consumer uncertainty increases, they prefer to extract now rather than later.

In addition, by increasing the rate of time preference ($\rho \uparrow$), the extraction rate of resources will decrease (c). The confidence in the future has increased with the preference rates increase. As a result, it is preferable to extract the resources in the future.

On the other hand, regarding (25) and the assumption that in the steady state, the growth rate of consumption and finished goods production is considered equal ($g_c = g_y = g^*$). Therefore: $g^* = [\rho + (2-\chi\kappa)r^* + (\beta + \chi\lambda)n - \chi\kappa(1-\chi\kappa)\sigma_R{}^2]\left[\frac{(1-\alpha-\chi\gamma)}{1-(\alpha+\chi\gamma)(2-\varsigma)}\right]$ (28)

According to equation (28), by increasing the rate of time preference ($\rho \uparrow$) as well as the population growth rate ($n \uparrow$), the economic growth rate in a steady state will increase ($g^* \uparrow$). However, with increasing uncertainty ($\sigma^2 \uparrow$), the economic growth rate in a steady state will decrease (cc).

6. Empirical Evidence for the Iranian Economy

The parameters of the finished goods and energy sector production functions are estimated in this section using data from the Iranian economy. The following is then calculated using these estimated parameters:

1) Optimal rate of energy resources extraction (r^*), (from equation 13)

2) The share of labor and capital in the finished goods and energy production sectors (η^*, θ^*), (in equations 10 and 11)

3) The optimum economic growth rate in a steady state (g^*), (equation 14).

It is worth noting that all of these rates are calculated under two different scenarios: certain and uncertain.

Table 1 summarizes the parameters and exogenous variables affecting the Iranian economy and the source of any data.

6.1 Calculation of the Fluctuation of Extractable Energy Resources

We use the Hodrick-Prescott Filter to determine the fluctuation of extractable energy resources. The Hodrick-Prescott Filter is a smoothing technique frequently used to obtain an approximate estimate of a series' long-term trend element. The technique was first used by (Hodrick & Prescott, 1997) to analyze postwar U.S. business cycles. After that, this filter has been used in (Pedersen, 2001), (Yamada, 2020), and (Sakarya & de Jong, 2020).

Technically, the Hodrick-Prescott (HP) filter is a two-sided linear filter that calculates the smoothed series st of yt by minimizing the variance of around, subject to the constraints of the second difference of st. the HP filter chooses st to minimize (29).

$$\sum_{t=1}^{T}(y_t - s_t)^2 + \lambda \sum_{t=2}^{T-1}[(s_{t+1} - s_t) - (s_t - s_{t-1})]^2 \quad (29)$$

The penalty parameter λ controls the smoothness of the series variance. The largerλ, the smoother the variance. As$\lambda \to \infty$, approaching a linear trend.

According to (Ravn & Uhlig, 2002), the value of λ is used as a power rule of 2 ($\lambda = 100$ for annual data, $\lambda = 1600$ for quarterly data, $\lambda = 14400$ for monthly data). EViews software was used to smooth the series of extractable energy resources (Rt) using the Hodrick-Prescott filter. Results for $\lambda = 100$ are shown in figure (1). Actual values, trends, and the growth rate of extractable energy resources (Rt) plus annual cycles are indicated in figure (1).

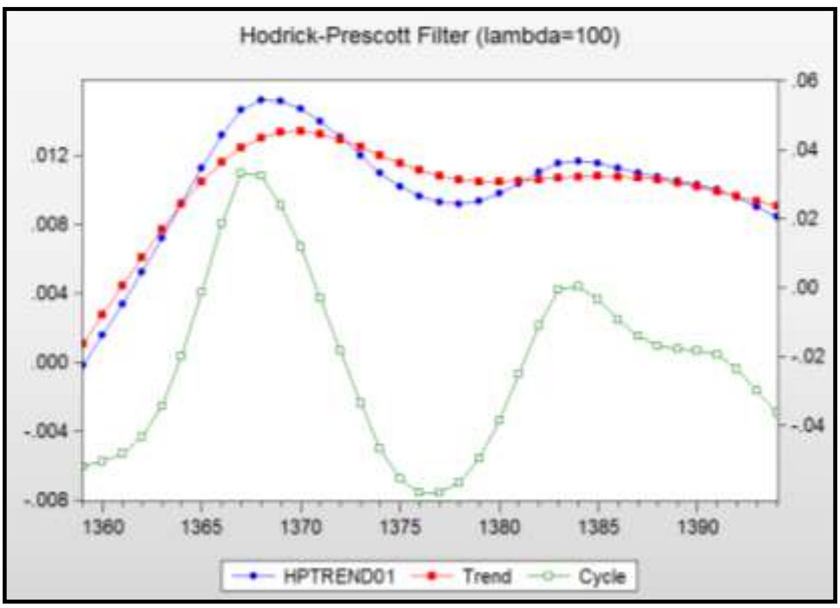

Figure 1. Annual cycles of actual values, trends, and growth rates of extractable energy resources are derived using Hodrick-Prescott filters.
***Source**: research findings*

Using Hodrick-Prescott filters and the variance of the data produced by this filter, the volatility of the Iranian economy's extractable energy resources growth rate was determined to be 0.22 from 1974 to 2014 ($\sigma^2 = 0.22$).

6.2 Calculating the Production Function Parameter

This section will calculate the parameters of the nonlinear production function discussed in equations (4 and 5). This study develops an alternative stochastic solution called the Genetic Algorithm (GA) approach for estimating the nonlinear production function.

$$Y_t = AB^{\chi} r^{\kappa\chi} \eta^{\alpha} (1-\eta)^{\gamma\chi} \theta^{\beta} (1-\theta)^{\lambda\chi} {K_t}^{\alpha+\gamma\chi} {H_t}^{\beta+\lambda\chi} {R_t}^{\kappa\chi}$$

A genetic algorithm (or GA) is a technique for solving optimization and search problems by calculating exact or approximate solutions. Genetic algorithms are regarded as heuristics for global search. Genetic algorithms are a subset of evolutionary algorithms that make use of evolutionary biology-inspired techniques such as inheritance, mutation, selection, and crossover (also called recombination)

The common form of nonlinear regression models can be written as:

$$Y_t = f(X_t, \theta) + \varepsilon_t \qquad (30)$$

Where X is an $(nx1)$ vector of independent variables, Y is the dependent variable, θ is a $(kx1)$ (nonlinear) parameter vector, and ε is a stochastic error.

According to (Ruelle, 2018), nonlinear models show economic realities better than linear patterns.

As in linear regression models, least squares or maximum likelihood methods are used in nonlinear regression models. In the least square method,

$$RSS = \sum_{i=1}^{n} \left[Y_i - f\left(X_i, \theta\right) \right]^2 \qquad (31)$$

We can obtain the parameters of the nonlinear regression model (θ) by minimizing RSS in equation (31). In comparison with linear models, analytical solution methods are not sufficient in solving the parameters of nonlinear models, and therefore, we need to employ iterative numerical search methods (Öztürk and Altan, 2008). In order to obtain the normal equations for the nonlinear regression model given by equation (30), we applied least-square standards by taking the derivative of RSS in equation (31) with respect to θ. ($\frac{\partial RSS}{\partial \theta_j} = 0 \qquad j = 1,2,\ldots,N$).

For a detailed study of the method of estimating the parameters of nonlinear patterns using a genetic algorithm, refer to (Eslamloueyan & Ostadzad, 2016). Table 1 summarizes the results of parameter estimation.

Table 1. Parameters and variables of the Iranian economy

Parameter or variable	Symbol	Value	Unit	Reference
Population growth rate	n	1.5	percent	Iran's Statistics Center
Inverse of intertemporal consumption elasticity of substitution	ς	0.79	-	Hadian and Ostadzad (2013)
Production elasticity than capital in final goods sector	α	0.49	-	The researchers' calculations
Production elasticity than labor in final goods sector	β	0.26	-	
Production elasticity than energy in final goods sector	χ	0.31		
Production elasticity than capital in energy-generation sector	γ	0.41	-	
Production elasticity than labor in energy-generation sector	λ	0.38	-	
Production elasticity than extractable energy resource in energy-generation sector	κ	0.3	-	
Share of labor force in final good's production sector	θ	70	percent	
Share of capital in final good's production sector	η	76	percent	
Rate of time preference	ρ	0.024	-	Eslamloueyan and Ostadzad (2014), (ABDOLI, 2009)
Depreciation rate	δ	0.037	percent	Amini and neshat (2005), (Ali Hossein Ostadzad & Behpour, 2015)

***Source**: research findings*

6.3 Model Calibration

This section discusses the parameters that were estimated or assumed to be calibrated by the Iranian economy. Economic growth rate (eq. 28) and the extraction rate (eq. 27) within the steady state and according to different amounts of extractable energy resource fluctuations (σ) are indicated in figures (2) and (3). Economic growth has been subjected to sensitivity analysis in light of the uncertainty surrounding exhaustible resources.

Empirical results show that with increasing energy resource volatility ($\sigma \uparrow$) and uncertainty, the economic growth rate will reduce ($g^* \downarrow$), and secondly, the extraction resource rate will also decrease ($r^* \downarrow$). Our results conform to some studies, including (Beladi, Deng & Hu, 2021), (Shaukat, Khan, Jafri & Hanif, 2019), (Levi, 2019); and (Thanopoulou & Strandenes, 2017). When economics is confronted with an uncertain future, firms appear to be more willing to innovate, and increased uncertainty can spur R&D (Banerjee & Siebert, 2017). Thus, businesses require human capital (or "skilled labor," that is, researchers). As a result of the increased demand for human capital, households begin to invest more time in education than in the finished goods sector. This promotes human capital accumulation in the economy, allowing firms to use human capital for research and development. This channel promotes technological advancement, thereby increasing growth and economic welfare, as long as uncertainty persists (Tsuboi, 2019). Simultaneously, as uncertainty increases, the standard negative effects of risk aversion start in and eventually supplant the former positive effect (Bekaert, Engstrom & Xu, 2019).

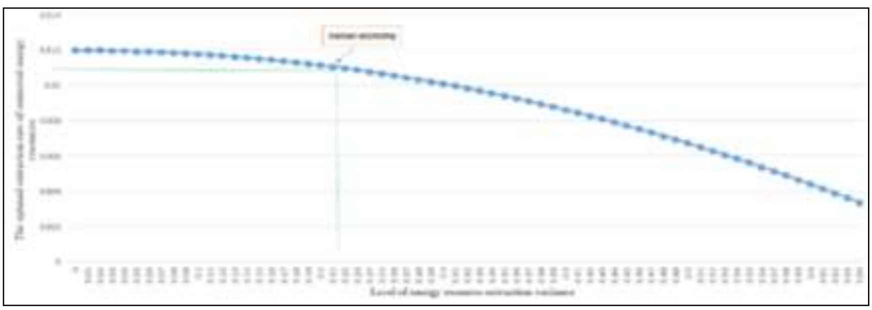

Figure 2. Optimal extraction rate of energy resources based on uncertainty in proven reserved resources
***Source**: research findings*

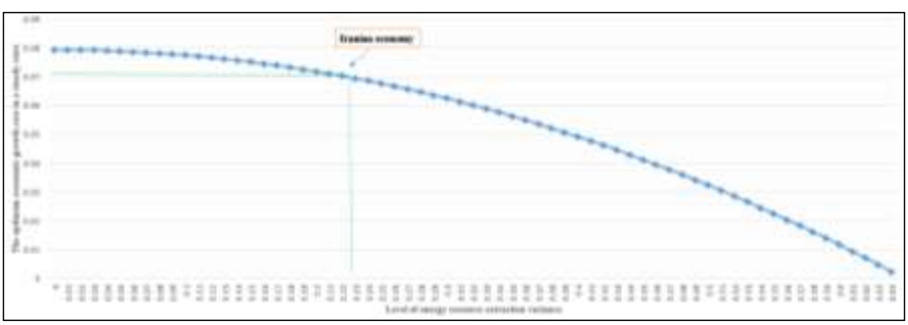

Figure 3. The optimal rate of economic growth based on the level of uncertainty in reserves
***Source**: research findings*

We estimated the volatility of energy resources in section 6.1 as approximately 0.22, so considering ($\sigma^2 = 0.22$) in fig (2) and (3), the economic growth rate in a steady state for the Iranian economy will be 7.1 percent. However, the extraction rate should be 1.1 percent to maintain a steady state of extractable energy resources. According to Figures 2 and 3, in the absence of fluctuations($\sigma^2 = 0$), the resource extraction rate will be 1.2 percent, and the optimum economic growth rate is 8 percent.

7. Conclusion

When attempting to study a complex system, scientists frequently apply stochastic modeling and analysis theories to obtain a system description, assuming that this will increase our knowledge and comprehension.

Uncertainty in exhaustible primary energy resources and their exploration can explain factors affecting economic growth and help achieve differences between countries with different growth rates. Therefore, it is helpful to study economic growth dynamics by considering the impact of stochastic non-renewable resources on economic growth. It can be verified in the form of a stochastic growth model.

Although several studies examined the influence of non-renewable primary energy resources on economic growth as a deterministic model, extractable random energy resources on economic growth are not considered in the stochastic growth model. In comparison to prior research, the most significant novelty in this study is:

1- Considering the stochastic extractable energy resources in a sustainable endogenous growth model.

2- Calculating the economic growth on a sustainable growth trajectory in a stochastic growth model in extractable primary energy sources than the deterministic extractable resource model.

We considered the random level of extractable resources into account and developed a random generalized growth model. Then, we solved the stochastic model analytically (using Stochastic Hamilton Jacobin Bellman method). We solved the model and then applied it to the Iranian economy. Finally, we conducted a sensitivity analysis on the fluctuation parameters.

The overall findings indicate that as energy resource uncertainty increases, economic growth declines, and the rate of resource extraction declines as well. It ultimately has the effect of slowing economic growth.

Due to the variance of energy extracting in the Iranian economy, estimated by our model ($\sigma^2 = 0.22$), the optimal economic growth rate in a steady state for this economy is 1.7 percent (in the case where neutral volatility and discounted intertemporal utility is maximized), and the rate of extraction should be 1.1 percent for neutralizing fluctuations of extractable energy resources in a steady state. According to figures 2 and 3, the resource extraction rate will be 1.2 percent in the absence of fluctuations($\sigma^2 = 0$), and the optimum economic growth rate is 8 percent.

Author Contributions

Conceptualization, methodology, validation, formal analysis, resources, writing—original draft preparation, writing—review and editing; all authors.

Funding
This research received no external funding.

Conflicts of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data used in the study were taken from:
https://www.amar.org.ir/english
https://tsd.cbi.ir/

Acknowledgements
Not applicable.

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